Freie Universität Berlin
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WS2015/16

Discrete Geometry III

Homework # 1 — due October 28th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **four** of your solutions. Only these will be graded. The problems marked with a ⧉ are **mandatory**. You can earn **20 points** on every (1 week) homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday. Homework sheets and additional information (including corrections) will be announced on the mailinglist. If you haven't signed up already, please do that at

`https://lists.fu-berlin.de/listinfo/dg3`

On the first exercise sheet, please state your full name, student id (Matrikel nummer), semester. and 'category' (e.g. Math, computer science; MSc, BSc, Diplom).

Exercise 1. Let $C_4(n)$ be the cyclic 4-polytope on n vertices.

i) Give a formula for $f_i(C_4(n))$ for all $0 \leq i \leq 3$.

ii) Explicitly compute the f-vectors of $C_4(5)$ and $C_4(9)$.

iii) Show that $\frac{1}{2}(f(C_4(5)) + f(C_4(9)))$ is not the f-vector of a 4-polytope.

(10 points)

Exercise 2. Let $P \subset \mathbb{R}^d$ be a simple d-polytope and $F \subseteq P$ a proper face. The **truncation** of F is a polytope $P' = P \cap \{x : \ell(x) \leq 0\}$ for an affine functional $\ell(x)$ such that $\ell(v) > 0$ for $v \in V(F)$ and $\ell(v) < 0$ for $v \in V(P) \setminus F$.

i) Show that P' is a simple polytope.

ii) For the case that F is a vertex, describe $h(P') - h(P)$.

iii) Bonus: Describe $h(P') - h(P)$ for general F.

(10+3 points)

Exercise 3. ⧉ Explicitly compute the h-vectors for the following classes of polytopes:

i) The d-dimensional **simplex** $\Delta_d = \text{conv}(e_1, \dots, e_{d+1})$;

ii) The d-dimensional **cube** $C_d = [0,1]^d$;

iii) For $k_1 \leq k_2 \leq \cdots \leq k_m$, the product of simplices

$$\Delta_{k_1,\dots,k_m} \ = \ \Delta_{k_1} \times \Delta_{k_2} \times \cdots \times \Delta_{k_m}$$

iv) In general, if P and Q are simple, then compute $h(P \times Q)$ from $h(P)$ and $h(Q)$.
[Hint: Think about this in terms of f-polynomials.]

(10 points)

Exercise 4. The $(d-1)$-dimensional permutahedron is the polytope

$$\Pi_{d-1} \ := \ \text{conv}\{(\pi(1),\ldots,\pi(d)) : \pi \text{ permutation}\}.$$

i) Let u, v be vertices of Π_{d-1}. Show that $[u,v] = \text{conv}(u,v)$ is an edge of Π_{d-1} if and only if $u - v = e_i - e_j$ for some $i \neq j$.
[Hint: Argue that it is sufficient to assume $v = (1, 2, \ldots, d)$].

ii) For a permutation $\pi = (\pi_1, \ldots, \pi_d)$ of $[d]$ an **ascent** is $1 \leq i < d$ such that $\pi_i < \pi_{i+1}$. Show that $h_k(\Pi_{d-1})$ is exactly the number of permutations with k ascents.
[Hint: Choose your linear function wisely.]

iii) Explicitly compute $h(\Pi_3) = (h_0, h_1, h_2, h_3)$.

(10 points)

Exercise 5. Let P be a 4-dimensional **simple** polytope:

i) Write out the f-vector equations obtained for $1 \leq k \leq 4$ from

$$\sum_{F \ k\text{-face of } P} \chi(F) = f_k(P),$$

where $\chi(F)$ is the Euler characteristic $\chi(F) = \sum_{i \geq 0} (-1)^i f_i(F)$.

ii) Show that these 4 equations give only 2 linearly independent conditions on f-vectors of simple polytopes.

(10 points)

Exercise 6. Let $\mathcal{J} \subseteq 2^{[n]}$ be a collection of subsets such that $A \cap B \in \mathcal{J}$ and $A \cup B \in \mathcal{J}$ for any $A, B \in \mathcal{J}$. For a field $\mathbf{k}$, let $\mathbf{k}[\mathcal{J}]$ be the $\mathbf{k}$-vector space with basis $\{e_A : A \in \mathcal{J}\}$.

i) Verify that $\mathbf{k}[\mathcal{J}]$ is a $\mathbf{k}$-algebra with multiplication given on generators by

$$e_A \cdot e_B \ := \ e_{A \cap B}$$

What is the unit in $\mathbf{k}[\mathcal{J}]$?

ii) Verify that $\mathbf{k}[\mathcal{J}]$ is also a $\mathbf{k}$-algebra with multiplication induced by

$$e_A \star e_B \ := \ \begin{cases} e_A & \text{if } A = B, \\ 0 & \text{otherwise.} \end{cases}$$

iii) Show that the linear map $\phi : \mathbf{k}[\mathcal{J}] \to \mathbf{k}[\mathcal{J}]$ given on basis elements by

$$\phi(e_A) \ := \sum_{B \in \mathcal{J}, B \subseteq A} e_B$$

gives a ring map $\phi : (\mathbf{k}[\mathcal{J}], \cdot) \to (\mathbf{k}[\mathcal{J}], \star)$. Show that this is even an isomorphism.

iv) [Bonus] Can you give an explicit inverse for ϕ?

(10+3 points)

Exercise 7. Let $n \geq 2$ be fixed. Let R be the standard graded $\mathbf{k}$-algebra with generators e_A for $\emptyset \neq A \subsetneq [n]$ and whose product satisfies

$$e_A e_B \ := \ 0 \quad \text{if and only if } A \not\subseteq B \text{ and } B \not\subseteq A.$$

For $n = 4$ compute $h(t)$ and d such that $F(R,t) = \frac{h(t)}{(1-t)^d}$. (Do you recognize the coefficients?)

(10 points)