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SS2013

Discrete Geometry II

Homework # 13— due July 10th

Exercise 1. In this exercise you will prove that every log-concave sequence arises as mixed volumes of two convex bodies. Let $T = \text{conv}\{e_0 := 0, e_1, \ldots, e_d\} \subset \mathbb{R}^d$ be a d-simplex.

i) Show that the following are the $d+1$ maximal cells of a triangulation of $T \times [0,1] \cong \mathrm{Cay}(T,T)$:

$$C_i \;=\; \text{conv}\{\tbinom{e_0}{0}, \tbinom{e_1}{0}, \ldots, \tbinom{e_i}{0}, \tbinom{e_i}{1}, \ldots, \tbinom{e_d}{1}\}$$

for $i = 0, 1, \ldots, d$.

For $\lambda \in \mathbb{R}^d$ such that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$ let $T^\lambda = \text{conv}\{e_0, \lambda_1 e_1, \ldots, \lambda_d e_d\}$ be a deformation of T.

ii) (Bonus) Show that the deformation C_i^λ of C_i yields a triangulation of $\mathrm{Cay}(T^\lambda, T)$.
[Hint: The crucial part is to show that $\bigcup_i C_i^\lambda = \mathrm{Cay}(T^\lambda, T)$.]

iii) Assuming ii) show that

$$MV(T^\lambda[k], T[d-k]) \;=\; \mathrm{vol}_d(T)\lambda_1\lambda_2\cdots\lambda_k$$

iv) Let $a_0, a_1, \ldots, a_d > 0$ be a log-concave sequence. Argue that we can assume that $a_0 = \frac{1}{d!}$ and show that for $\lambda_1 = \frac{a_0}{a_1}$ and $\lambda_i = \frac{a_{i-1}a_{i+1}}{a_i^2}$ for $i \ge 2$ we can use iii) to get $a_i \;=\; MV(T^\lambda[k], T[d-k])$.

(10+3 points)

Exercise 2. i) Let $\omega_k = \mathrm{vol}_k(B_k)$ be the volume of the k-dimensional unit ball. Show that the sequence $(\omega_0, \omega_1, \omega_2, \ldots)$ is log-concave.

ii) Show that the sequence $a_k = \binom{d}{k}$, $k = 0, \ldots, d$, is log-concave but not concave.

(10 points)

Exercise 3. i) Let $S = [p,q] \subset \mathbb{R}^d$ be a lattice segment (i.e. $p, q \in \mathbb{Z}^d$). Show that $E_S(n) = g \cdot n + 1$ where g is the greatest common divisor of the numbers $|p_i - q_i|$ for $i = 1, \ldots, d$.

Let $P \subset \mathbb{R}^2$ be a lattice triangle.

i) P is called *special* if P has two sides parallel to the coordinate axes. Show that $E_P(n)$ is a polynomial.
[Hint: You can complete P to an axes-aligned rectangle.]

ii) Show that $E_P(n)$ is a polynomial for arbitrary lattice triangles.
[Hint: Show that you can trap P in an axis-aligned rectangle.]

(10 points)