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Discrete Geometry II

Homework # 12— due July 3rd

Exercise 1. Let $P, Q \subset \mathbb{R}^2$ be two polygons in the plane. Minkowski's first inequality implies that mixed volume (here also called mixed area) satisfies

$$MV(P,Q)^2 \;\geq\; \mathrm{vol}_2(P)\,\mathrm{vol}_2(Q)$$

and with equality if and only if P and Q are homothetic. The goal of this exercise is to prove the inequality in the plane.

i) Show that it is sufficient to assume that $\mathrm{vol}_2(P) = \mathrm{vol}_2(Q) = 1$.

ii) Let $a_1, \dots, a_n \in \mathbb{R}^2$ be the unit facet normals of P and define

$$Q' \;:=\; \{x \in \mathbb{R}^2 : a_i^t x \leq h_Q(a_i) \text{ for } i = 1, \dots, n\}.$$

Show that $\mathrm{vol}_2(Q') \geq 1$ and show that $MV(Q,P) \geq MV(Q',P)$.
[Hint: Consider the support function of Q'.]

iii) Show that $MV(Q',P) \geq 1$.
[Hint: P is a solution to a Minkowski problem.]

(10 points)

Exercise 2. Denote by $\pi_i \colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ the orthogonal projection that deletes the i-th coordinate and let $K \subset \mathbb{R}^d$ be a convex body.

i) Show that

$$\frac{1}{d}\sum_{i=1}^{d} \mathrm{vol}_{d-1}(\pi_i(K)) \;=\; \mathrm{MV}(K[d-1], [0,1]^d).$$

ii) Prove the inequality

$$\frac{1}{d}\sum_{i=1}^{d} \mathrm{vol}_{d-1}(\pi_i(K)) \;\geq\; \mathrm{vol}_d(K)^{\frac{d-1}{d}}$$

and give an example of K for which equality holds.

(10 points)

Exercise 3. Let $a_1, \dots, a_n \in \mathbb{R}^2$ be positively spanning unit vectors and assume that they are ordered counterclockwise. Let $\mathcal{B} \subset \mathbb{R}^n$ be the set of those $b \in \mathbb{R}^n$ for which $P(b) = \{x \in \mathbb{R}^2 : a_i^t x \leq b_i \text{ for } i = 1, \dots, n\}$ is a polygon and all hyperplanes $\{x : a_i^t x = b_i\}$ are supporting.

i) Show that there is a quadratic form $q(y) = \frac{1}{2} y^t Q y$ with $Q \in \mathbb{R}^{n \times n}$ symmetric such that $\mathrm{vol}_2(P(b)) = q(b)$ for all $b \in \mathcal{B}$.

ii) Show that the form $q(b)$ has nullity ≥ 2 and Q has at least one positive eigenvalue. (The nullity of $q(x)$ is the dimension of $\ker(Q)$.)

iii) Show that

$$MV(P(b), P(b')) \;=\; b^t Q b'$$

for $b, b' \in \mathcal{B}$.

iv) (Bonus) With the help of the Minkowski inequality

$$q(b,b')^2 \;\geq\; q(b,b)q(b',b')$$

prove that Q has exactly one positive eigenvalue.

(10+3 points)