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SS2013

Discrete Geometry II

Homework # 11— due June 26th

Exercise 1. For a vector $u \in \mathbb{R}^d \setminus \{0\}$ let us write $[0,u] = \{\lambda u : 0 \le \lambda \le 1\}$ and let us denote by $u^\perp = \{x \in \mathbb{R}^d : u^t x = 0\}$ the hyperplane perpendicular to u.

i) Show that for a convex body $K \subset \mathbb{R}^d$

$$\mathrm{MV}(K[d-1],[0,u]) \;=\; \frac{\|u\|}{d}\,\mathrm{vol}_{d-1}(\pi_u(K))$$

where $\pi_u : \mathbb{R}^d \to u^\perp$ is the orthogonal projection onto $u^\perp$.

ii) Show that the function $f : \mathbb{R}^d \to \mathbb{R}$ given by $f(u) := \mathrm{MV}(K[d-1],[0,u])$ is convex.
[Hint: First observe that f is positively linear and then show that $f(u+v) \le f(u) + f(v)$.]

(10 points)

Exercise 2. i) Let $\Delta \subset \mathbb{R}^d$ be a d-simplex and let $P \subset \mathbb{R}^d$ be the result of truncating a vertex of Δ. Show that

$$\mathrm{MV}(\Delta[d-i], P[i]) \;=\; \mathrm{vol}(\Delta)$$

for all $d > i \ge 0$. Infer that the mixed volume is not *strictly* monotone with respect to inclusion.

ii) Let $K, L \subset \mathbb{R}^d$ be convex bodies and let $d \ge r > \dim K$. Show that

$$\mathrm{MV}(K[r], L[d-r]) \;=\; 0.$$

(10 points)

Exercise 3. i) Let $z_1, z_2, \ldots, z_n \in \mathbb{R}^d \setminus \{0\}$ and define the zonotope

$$Z \;:=\; [0,z_1] + [0,z_2] + \cdots + [0,z_n]$$

Show that

$$\mathrm{vol}_d(Z) \;=\; \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} |\det(z_{i_1}, z_{i_2}, \ldots, z_{i_d})|$$

ii) Consider the function $f(u) := \mathrm{vol}_{d-1}(\pi_u(Z))$ where $\pi_u : \mathbb{R}^d \to u^\perp \cong \mathbb{R}^{d-1}$ is defined in Exercise 1. Show that there is a zonotope $\Pi_Z = \sum_{j=1}^M [-w_j, w_j]$ such that

$$f(u) \;=\; \max\{u^t x : x \in \Pi_Z\}.$$

(10 points)