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Discrete Geometry II

Homework # 10— due June 20th

Exercise 1. Let $A \in \mathbb{R}^{n\times d}$ be a matrix of rank $d < n$ with positively dependent rows (i.e., $\alpha^t A = 0$ for some $\alpha \in \mathbb{R}^n_{>0}$). Let $\bar{A} = (\bar{a}_1, \dots, \bar{a}_n) \in \mathbb{R}^{(n-d)\times n}$ be of rank $n-d$ and $\bar{A}A = 0$. For $b \in \mathbb{R}^n$ set $\bar{b} := \pi(b) = \bar{A}b$.

i) Show that $P_A(b) = \{x \in \mathbb{R}^d : Ax \le b\}$ is affinely isomorphic to $\{y \in \mathbb{R}^n : y \ge 0, \bar{A}y = \bar{b}\}$.

Now let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

ii) Compute $\pi(\mathcal{B}_A)$ and the closed inner region $\bar{\mathcal{B}}_A$. These are two pointed 3-dimensional cones. Draw a corresponding 2-dimensional picture.
[Hint: This are questions about a planar point configuration.]

iii) (Bonus) How many normally non-equivalent types of simple polytopes with 6 facets are there for A?

(10 points)

Exercise 2. Let $P, Q \subset \mathbb{R}^d$ be polytopes. The image of $P \times Q \subset \mathbb{R}^d \times \mathbb{R}^d$ under the linear projection $\pi(x, y) = x + y$ is the Minkowski sum $P + Q$. Let $c \in \mathbb{R}^d \times \mathbb{R}^d$ be a fixed direction.
For every $p \in P + Q$ the fiber $F_p := \pi^{-1}(p) \cap (P \times Q)$ is a polytope and $F_p^c = \{x \in F_p : c^t p \text{ maximal}\}$ is a face of F_p.

i) Show that for every $p \in P+Q$ there is a unique minimal face $G_p \subseteq P \times Q$ such that $F_p^c = G_p \cap \pi^{-1}(p)$.

Denote by $\mathcal{L} := \{G_p : p \in P + Q\}$ the set of these faces.

ii) Show that $\mathcal{K} = \{\pi(G_p) : G_p \in \mathcal{L}\}$ is a mixed subdivision of $P + Q$.
[Hint: Show first that $\mathcal{L}$ is a polyhedral complex.]

iii) Show $\mathcal{K}$ is an exact mixed subdivision if F_p^c is a vertex for all $p \in P + Q$.

(10 points)

Exercise 3. Let $P_1, \dots, P_d \subset \mathbb{R}^d$ such that $0 \in \operatorname{relint}(P_i)$ for all i and $P = P_1 + \cdots + P_d$ is of dimension d. Show that the following statements are equivalent.

i) There is an exact mixed subdivision of P with a cell F of type $d(F) = (1, 1, \dots, 1)$.

ii) There are $u_i \in P_i$ such that $u_1, \dots, u_d$ are linearly independent.

iii) For every $I \subseteq [d]$

$$\dim \sum_{i\in I} P_i \ \ge\ |I|.$$

(10 points)