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SS2013

Discrete Geometry II

Homework # 9— due June 12th

Exercise 1. Consider the following five unit vectors in $\mathbb{R}^3$:

$$a_1 = \frac{1}{\sqrt{3}}(1,1,1), \quad a_2 = \frac{1}{\sqrt{3}}(1,-1,1), \quad a_3 = \frac{1}{\sqrt{3}}(-1,-1,1),$$
$$a_4 = \frac{1}{\sqrt{3}}(-1,1,1), \quad a_5 = (0,0,-1)$$

i) Describe the cone $\mathcal{C} := \mathcal{B}_A \cap \{b \in \mathbb{R}^5 \mid b_3 = b_4 = b_5 = 0\}$.

ii) Express the volume $\mathrm{vol}(P_A(b))$ as a piecewise polynomial function of the variables b_1 and b_2 on $\mathcal{C}$.

(10 points)

Exercise 2. i) Give an example of two polytopes $P_A(b')$ and $P_A(b'')$ such that

$$P_A(b') + P_A(b'') \neq P_A(b' + b'')$$

ii) Consider the following five unit vectors in $\mathbb{R}^3$:

$$a_1 = (-1,0,0), \quad a_2 = (0,-1,0), \quad a_3 = (0,0,-1),$$
$$a_4 = \frac{1}{\sqrt{3}}(1,1,1), \quad a_5 = (a,b,c) \in \mathbb{R}^3_{<0}$$

Find the system of linear inequalities on b_4 and b_5 that describes the cone $\mathcal{B}_A \cap \{b \in \mathbb{R}^5 \mid b_1 = b_2 = b_3 = 0\}$.

(10 points)

Exercise 3. Let $a_1, \ldots, a_n \in \mathbb{R}^d_{>0}$ have strictly positive coordinates. For $b \in \mathbb{R}^n_{\geq 0}$ define the *orthant-shaped* polyhedron

$$Q(b) \;:=\; \{x \in \mathbb{R}^d : x \geq 0,\, a_i^t x \geq b_i \text{ for all } i = 1, 2, \ldots, n\}.$$

For $\alpha \in \mathbb{R}^n_{>0}$ define

$$\mathcal{Q} \;:=\; \{b \in \mathbb{R}^n_{\geq 0} : \mathrm{vol}_{d-1}\, Q(b)^{a_i} \leq \alpha_i \text{ for all } i = 1, \ldots, n\}$$

i) For $i \in [n]$ and $\epsilon > 0$ define $\bar{b} = b + \epsilon e_i$ and show that

$$\mathrm{vol}_{d-1}(Q(b)^{a_i}) \;\leq\; \mathrm{vol}_{d-1}(Q(\bar{b})^{a_i})$$

ii) Show that there is an M such that $b_i \leq M$ for all $b \in \mathcal{Q}$.

iii) Deduce that there is a $U > 0$ such that

$$P(b) \;:=\; Q(b) \cap \{x : x_1 + x_2 + \cdots + x_d \leq U\}$$

is a polytope such that $P(b)^{a_i} = Q(b)^{a_i}$ for all i.

iv) (Bonus) Show that $\mathcal{Q}$ is convex.

(10+3 points)

Supplementary problems

Exercise 4. For $d+1$ positively dependent vectors $a_0, a_1, \ldots, a_d \in \mathbb{R}^d$ and a vector $b \in \mathbb{R}^{d+1}$ compute the volume of the simplex $P_A(b)$.

Exercise 5. Let $A = (a_1, \ldots, a_n)$ be a configuration of unit vectors in $\mathbb{R}^d$. Define the *contraction* $c_i(A)$ as the following configuration of $n-1$ vectors in $a_i^\perp$:

$$c_i(a_j) = a_j - (a_i^\top a_j)a_i \text{ for all } j \neq i$$

i) Show that the face $F_i(b)$ of the polyhedron $P_A(b)$ is congruent to the polyhedron $P_{c_i(A)}(d_i(b)) \cap a^\perp$, where $d_i(b) = (b_1, \ldots, \hat{b_i}, \ldots, b_n) \in \mathbb{R}^{n-1}$.

ii) Prove the formulas

$$\frac{\partial \operatorname{vol}_{d-1}(F_i(b))}{\partial b_j} = \frac{\operatorname{vol}_{d-2}(F_{ij}(b))}{\sin \phi_{ij}}$$

$$\frac{\partial \operatorname{vol}_{d-1}(F_i(b))}{\partial b_i} = -\sum_{j\neq i} \operatorname{vol}_{d-2}(F_{ij}(b)) \cot \phi_{ij}$$

where $F_{ij}(b) = F_i(b) \cap F_j(b)$, and ϕ_{ij} is the angle between a_i and a_j.

iii) (Bonus) Show that the volume is twice continuously differentiable on a subset of $\mathcal{B}_A$ consisting of all b such that the system $Ax \leq b$ is irredundant. Give an example showing that the volume is not thrice differentiable.