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Discrete Geometry II

Homework # 8 — due June 5th

Exercise 1. Let $K_1, \ldots, K_n \subset \mathbb{R}^d$ be convex bodies. The *Cayley embedding* of $\{K_i\}_i$ is

$$C = \mathrm{Cay}(K_1, K_2, \ldots, K_n) \;:=\; \text{conv}\{(p_i, e_i) : p_i \in K_i, i = 1, 2, \ldots, n\} \subset \mathbb{R}^d \times \mathbb{R}^n$$

i) For $\lambda \in \mathbb{R}^n$ such that $\lambda \geq 0$ and $\sum_i \lambda_i = 1$ show that

$$C \cap \{(x, y) : y = \lambda\} \;\cong\; \sum_{i=1}^{n} \lambda_i K_i$$

ii) Show that for every proper subset $I \subset [n]$ the Cayley embedding $\mathrm{Cay}(K_i : i \in I)$ is a face of C.

iii) Let $P \subset \mathbb{R}^M$ be a polytope and let $\pi : \mathbb{R}^M \to \mathbb{R}^n$ be a linear projection. Show that if $Q = \pi(P)$ is an $(n-1)$-simplex and every vertex of P is mapped to a vertex of Q, then P is linearly isomorphic to $\mathrm{Cay}(P_1, \ldots, P_n)$ for some polytopes $P_1, \ldots, P_n$.

(10 points)

Exercise 2. For a d-dimensional polytope $P \subset \mathbb{R}^d$ define

$$f_P(t) := \mathrm{vol}_d(P + tB_d)$$

where B_d is the unit ball.

i) For $d = 2$ show that $f_P(t)$ is a polynomial of degree 2 and give an interpretation for the coefficients.

ii) Show that the Brunn-Minkowski inequality for $P \subset \mathbb{R}^2$ and B_2 is equivalent to the isoperimetric inequality $p^2 \geq 4\pi a$, where p is the perimeter, and a is the area of P.

iii) For $d = 3$ show that $f_P(t)$ is a polynomial of degree 3 and interpret its coefficients.

(10+3 points)

Exercise 3. i) Prove the *weighted* arithmetic-geometric-mean inequality

$$\sum_{i=1}^{n} \lambda_i z_i \;\geq\; \prod_{i=1}^{n} z_i^{\lambda_i}$$

for all $z_i \geq 0$ and $\lambda_i \in [0, 1]$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

ii) Prove the *multiplicative* Brunn-Minkowski inequality:

$$\mathrm{vol}_d((1-\lambda)K + \lambda L) \;\geq\; \mathrm{vol}_d(K)^{1-\lambda} \cdot \mathrm{vol}_d(L)^{\lambda}$$

for all convex bodies $K, L \subset \mathbb{R}^d$ and $\lambda \in [0, 1]$.

iii) What are the equality cases?

(10 points)