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# Discrete Geometry II

## Homework # 7— due May 29th

This time, you are required to choose two problems $i, j$ such that $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

**Exercise 1.** Let $\Delta = \text{conv}\{v_0, v_1, \ldots, v_d\} \subset B_d$ be a $d$-simplex inscribed in the unit ball. For every $1 \leq i \leq d+1$, let $F_i$ be the facet not containing $v_i$.

i) A simplex $\Delta$ is called *regular* if every edge has the same length. Show that if $\Delta$ has maximal volume, then $\Delta$ is regular.
[Hint: Show that $\Delta$ has maximal volume if and only if for every $i$ the hyperplane $H_i$ through $v_i$ parallel to $F_i$ is supporting for $B_d$.]

From now on assume that $\Delta$ has maximal volume among all simplices with vertices in $B_d$.

ii) Compute the volume of $\Delta$ and the radius of the largest ball contained in $\Delta$.

**(10 points)**

**Exercise 2.**

i) Compute the volume of the $d$-dimensional regular simplex with edge length 1.

ii) Let $B(v_i, \frac{1}{2})$, $i = 0, \ldots, d$, be pairwise tangent balls centered at the vertices of a regular simplex with edge length 1. Let $B$ be the ball tangent to all $B(v_i, \frac{1}{2})$. Compute the radius of $B$.

**(10 points)**

**Exercise 3.** Let $\mathcal{E} = \mathcal{E}(A, 0)$ for some positive definite matrix $A$.

i) Show that the polar of $\mathcal{E}$ is also an ellipsoid. That is, show that there is a positive definite matrix $A'$ such that $\mathcal{E}^{\triangle} = \mathcal{E}(A', 0)$.

ii) Let $K \subset B_d$ be a $d$-dimensional convex body with $-K = K$. Show that the set $\{A \in \mathrm{PSD}_d : K \subseteq \mathcal{E}(A, 0), \mathrm{vol}(\mathcal{E}(A, 0)) \leq \mathrm{vol}(B_d)\}$ is closed and bounded.

iii) Infer that there exists an ellipsoid of minimum volume containing $K$ and an ellipsoid of maximum volume contained in $K$.

**(10 points)**

**Exercise 4.** Let $a, b, c \in \mathbb{R}^2$ be three non-collinear points.

i) Show that there is a unique ellipse $\mathcal{E}$ inscribed in the triangle $T = \text{conv}\{a, b, c\}$ and tangent to its sides at their midpoints.

ii) Show that the area of $\mathcal{E}$ is $\frac{\pi}{3\sqrt{3}}$ times the area of the triangle $T$ and that the area of any other inscribed ellipse is smaller than that.

iii) (Bonus) Identifying $\mathbb{R}^2$ with $\mathbb{C}$, put $P(z) = (z-a)(z-b)(z-c)$. Show that the foci of the ellipse $\mathcal{E}$ are the zeros of the derivative $P'(z)$.

**(10+3 points)**