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SS2013

Discrete Geometry II

Homework # 6— due May 22nd

Exercise 1. Let $C_d = [0,1]^d$ be the d-cube. Define $\overline{\omega} : C \to \mathbb{R}$ by

$$\overline{\omega}(p) := \sum_{1 \le i < j \le d} |p_i - p_j|$$

for $p \in C_d$. Show that $\overline{\omega}$ is a piecewise-linear convex function and determine the domains of linearity, that is, determine the maximal cells in the induced subdivision $\mathcal{K}^\omega$ (where ω is the restriction of $\overline{\omega}$ to the vertices of C_d).

(10 points)

Exercise 2. Let $P \subset \mathbb{R}^d$ be a d-polytope and let $b_F \in \operatorname{relint}(F)$ for all non-empty faces $F \subseteq P$. For $\varepsilon > 0$ define

$$\omega(b_F) \;:=\; \varepsilon^{\dim(F)}$$

i) Show that if ε is sufficiently small, then

$$\operatorname{conv}\{\tbinom{b_F}{\omega(b_F)}, \tbinom{b_P}{\omega(b_P)}\}$$

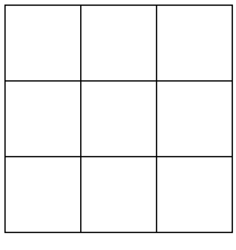
is an edge of P^ω for every proper face $F \subset P$.
[Hint: Writing any point on the segment as a convex combination of other points will have larger height.]

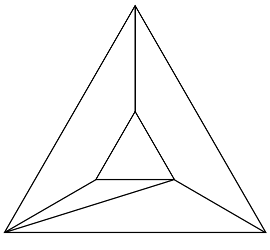
ii) Infer that the maximal cells of $\mathcal{K}^\omega(P)$ are pyramids over the maximal cells of $\mathcal{K}^\omega(F)$ for F a facet of P.

iii) Show that the regular subdivision $\mathcal{K}^\omega$ is the barycentric subdivision.
[Hint: Induction on the dimension.]

(10 points)

Exercise 3. For each of the shown subdivisions, determine whether it is regular or not.





(10 points)

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Exercise 4. Let $s_1, s_2, s_3, s_4 \in \mathbb{R}^2$ be the vertices of a convex quadrilateral P. There are three distinct subdivisions of P: one consisting of P only, and two where P is subdivided by a diagonal. A subdivision of P is Delaunay if every cell C has a circumcircle and all vertices outside C lie outside its circumcircle.

Prove the following two characterizations of the Delaunay subdivision.

i) Show that for $\omega(s) := \|s\|^2$, the subdivision $\mathcal{K}^\omega$ of P is Delaunay.
[Hint: Changing ω by a linear function does not change $\mathcal{K}^\omega$.]

ii) Identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$. Let $S \subset \mathbb{R}^3$ be the unit sphere centered at $(0,0,1)$ and $a = (0,0,2)$ be its north pole. Denote by $\widehat{s_i}$ the intersection $S \cap (a, s_i]$ (stereographic projection). Then the Delaunay subdivision consists of the faces of conv$\{a, \widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \widehat{s_4}\}$ projected from the point a back to $\mathbb{R}^2$.

(10 points)