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SS2013

Discrete Geometry II

Homework # 5— due May 15th

Exercise 1. i) Let P be a d-polytope with facets $F_1, F_2, \dots, F_m$. Show that

$$\mathrm{vol}_{d-1}(F_1) \; < \; \sum_{i=2}^{m} \mathrm{vol}_{d-1}(F_i)$$

ii) Prove Minkowski's theorem in the plane: If $u_1, \dots, u_m \in \mathbb{R}^2$ are unit vectors spanning the plane and $\alpha_1, \dots, \alpha_m > 0$ such that $\alpha_1 u_1 + \cdots + \alpha_m u_m = 0$, then there is a polygon P, unique up to translation, with outer facet normals u_i and corresponding facet volumes α_i.

(10 points)

Exercise 2. Show that the barycentric subdivision $\tilde{\mathrm{sd}}(P)$ gives a dissection of P into simplices. For that you have to show that $\mathrm{int}\,\Delta(\mathcal{F}_1) \cap \mathrm{int}\,\Delta(\mathcal{F}_2) = \varnothing$ for any two flags $\mathcal{F}_1 \neq \mathcal{F}_2$ and that for every point $p \in P$ there is a flag $\mathcal{F}$ with $p \in \Delta(\mathcal{F})$.

(10 points)

Exercise 3. Let $P = \mathrm{conv}\{v_0, v_1, \dots, v_d\}$ be a d-simplex. Consider the matrix

$$D \;=\; \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & \ell_{00} & \ell_{01} & \cdots & \ell_{0d} \\ 1 & \ell_{10} & \ddots & & \ell_{1d} \\ \vdots & \vdots & & & \vdots \\ 1 & \ell_{d0} & \ell_{d1} & \cdots & \ell_{dd} \end{pmatrix} \qquad \text{where } \ell_{ij} = \|v_i - v_j\|^2.$$

Show that $2^d (d!)^2 \, \mathrm{vol}_d(P)^2 = (-1)^{d-1} \det(D)$.

(10 points)

Exercise 4. For a triangle $\Delta = \mathrm{conv}\{a, b, c\}$ with ordered vertices $a, b, c \in \mathbb{R}^2$, let us define the *signed volume* of Δ as

$$\mathrm{vol}^{\circ}(\Delta) \;=\; \frac{1}{2} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \end{pmatrix}.$$

Let $P = \mathrm{conv}\{v_1, \dots, v_n\}$ be a polygon with vertices ordered counterclockwise, and let $p \in \mathrm{int}(P)$ be a point in the interior.

i) Show that

$$(\star) \qquad \mathrm{vol}_2(P) \;=\; \frac{1}{2} \sum_{i=1}^{n} \det \begin{pmatrix} 1 & 1 & 1 \\ p & v_i & v_{i+1} \end{pmatrix} \qquad \text{where } v_{n+1} := v_1.$$

ii) Argue geometrically that the right-hand side of $(\star)$ is independent of the choice of $p \in \mathbb{R}^2$ (whether inside P or not). This shows that the volume is a polynomial in the vertex coordinates, invariant under translation.

(10 points)