Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Ivan Izmestiev

SS2013

Discrete Geometry II

Homework # 4— due May 8th

Exercise 1.

i) Let $V = \{v_0, v_1, \dots, v_d\} \subset \mathbb{R}^d$. Show that $V \subseteq H$ for some hyperplane H if and only if

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \cdots & v_d \end{pmatrix} = 0.$$

ii) Let $P = \text{conv}\{v_1, v_2, \dots, v_n\}$ be a polytope and let $\epsilon > 0$. Show that there is a simplicial polytope $P' = \text{conv}\{v'_1, v'_2, \dots, v'_m\}$ with Hausdorff distance $d(P, P') \leq \epsilon$.

iii) Let $P_1 = \text{conv}\{\pm e_1, \pm e_2\}$. What is the square $P_2 \subset \mathbb{R}^2$ with vertices on the unit circle and $d(P_1, P_2)$ maximal?

(10 points)

Exercise 2. For a box $B = \{x \in \mathbb{R}^d : a_i \leq x_i \leq b_i \text{ for } i = 1, \dots, d\}$ define the volume of B as $\text{vol}_d(B) := (b_1 - a_1) \cdots (b_d - a_d)$ if $B \neq \varnothing$ and $\text{vol}_d(\varnothing) := 0$.

i) Show that vol_d is a *valuation* on the family of boxes: If B, C are boxes such that $B \cup C$ is a box, then

$$\text{vol}_d(B \cup C) + \text{vol}_d(B \cap C) = \text{vol}_d(B) + \text{vol}_d(C).$$

ii) Let $B, B_1, \dots, B_k$ be boxes such that $B = B_1 \cup B_2 \cup \cdots \cup B_k$ and $\text{int}(B_i) \cap \text{int}(B_j) = \varnothing$ for all $i \neq j$. Show that

$$\text{vol}_d(B) = \text{vol}_d(B_1) + \cdots + \text{vol}_d(B_k).$$

[Hint: Pick a suitable hyperplane H and consider $B \cap H^+$ and $B \cap H^-$.]

iii) Deduce that vol_d is well-defined on *polyboxes*: For a polybox $S \subset \mathbb{R}^d$, we can set

$$\text{vol}_d(S) := \text{vol}_d(B_1) + \cdots + \text{vol}_d(B_k)$$

for any collection $B_1, \dots, B_k$ of boxes such that $S = B_1 \cup B_2 \cup \cdots \cup B_k$ and $\text{int}(B_i) \cap \text{int}(B_j) = \varnothing$ for all $i \neq j$.
[Hint: For two such representations consider the 'common refinement'.]

(10 points)

Exercise 3. Let K, L be convex bodies in $\mathbb{R}^d$ such that $0 \in \text{int}(K)$ and $0 \in \text{int}(L)$.

i) Prove or disprove: For any $\epsilon > 0$ there exists $\delta > 0$ such that $d(K, L) < \delta$ implies $d(K^\triangle, L^\triangle) < \epsilon$.

ii) Show that if $|\frac{1}{d_K}(x) - \frac{1}{d_L}(x)| < \epsilon$ for all $x \in \mathbb{S}^{d-1}$, then $d(K, L) < \epsilon$.

iii) Assume additionally that both K and L are contained in $B_R(0)$. Show that then $|d_K(x) - d_L(x)| < \frac{\epsilon}{R^2}$ for all $x \in \mathbb{S}^{d-1}$ implies $d(K, L) < \epsilon$.

(10 points)