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SS2013

Discrete Geometry II

Homework # 3— due May 2nd

As May 1st is a holiday, we ask you to hand in your solutions on **May 2nd**. As usual you are strongly encouraged to solve all the problems and you are supposed to hand in solutions for only two problems. This time, however, you are required to choose two problems i, j such that $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$.

Exercise 1. Let $\{K_i : i \in I\}$ be a family of convex bodies in $\mathbb{R}^d$. Let $1 \leq n \leq d+1$ be fixed and suppose that any set of n of these convex bodies has a non-empty intersection. Show that for every linear subspace $U \subseteq \mathbb{R}^d$ of dimension $d+1-n$ there is a $t \in \mathbb{R}^d$ such that $(t+U) \cap K_i \neq \emptyset$ for all $i \in I$.

(10 points)

Exercise 2. Let $A_1, A_2, \dots, A_n$ and C be convex subsets of $\mathbb{R}^d$. Assume that for any $I \subseteq [n]$ with $|I| = d+1$ there is a $t \in \mathbb{R}^d$ such that

$$t + C \ \subseteq \ \bigcap_{i \in I} A_i.$$

Show that there is a $t \in \mathbb{R}^d$ such that $t + C \subseteq A_i$ for all $i = 1, 2, \dots, n$.

(10 points)

Exercise 3. Show that from every finite covering of $\mathbb{R}^d$ by open half-spaces one can choose a subcovering consisting of not more than $d+1$ half-spaces.

(10 points)

Exercise 4. Show that the points $v_1, \dots, v_n \in \mathbb{R}^d$ are in convex position (i. e. each of them is a vertex of conv$\{v_1, \dots, v_n\}$) if and only if the system of linear inequalities $v_i^\top x \leq 1$ is irredundant.

(10 points)

Exercise 5. Consider the following polytope in $\mathbb{R}^4$:

$$P := \text{conv}\{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\},$$

where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$.

i) Show that $P^\triangle \supseteq Q$ and $P \subseteq Q^\triangle$, where Q is the convex hull of the 24 points

$$\pm e_1, \pm e_2, \pm e_3, \pm e_4, \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

ii) Show that $P^\triangle = Q$.

iii) (Bonus) Show that $P^\triangle$ is congruent to a scaled copy of P.

The polytope P is called 24-cell.

(10+3 points)

Supplementary problems

Exercise 6. Let $S \subset \mathbb{R}^d$ be a compact set such that for every $d+1$ points $x_1, \ldots, x_{d+1} \in S$ there is a point $p \in S$ such that $[p, x_i] \subset S$ for all i. Show that then S is star-shaped, that is there exists $s \in S$ such that $[s, x] \subset S$ for all $x \in S$.