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Discrete Geometry II

Homework # 1— due April 17th

Please work in **pairs** on the homeworks. (If you don't have a partner, contact Ivan). Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. The 'supplementary problems' are extra. They do not count as homework problems. The problems marked with a ■ are **mandatory**. Please mark **two** of your solutions. Only these will be graded. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday.
On the first exercise sheet, please state your full name, student id (Matrikel nummer), semester. and 'category' (e.g. Math, computer science; MSc, BSc, Diplom). Please also 'solve' the first problem!

Exercise 1. ■ Sign up on the **mailing list** at

`https://lists.fu-berlin.de/listinfo/DG2`

(0 points)

Exercise 2. ■ Let $A, B \subseteq \mathbb{R}^d$ be convex sets. Show that the *Minkowski sum*

$$A + B \;:=\; \{a + b : a \in A, b \in B\}$$

is a convex set.

(10 points)

Exercise 3. A set $X \subseteq \mathbb{R}^d$ is called *path-connected* if for every two $p_0, p_1 \in X$ there is a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.

Let $K \subset \mathbb{R}^d$ be a (full-dimensional) convex body. Show that if the set of extreme points $X = \operatorname{ext}(K)$ is path-connected, then every point $p \in K$ is in the convex hull of d points. (This improves Carathéodory's theorem by 1.)

(10 points)

Exercise 4. Let $C \subset \mathbb{R}^d$ be a compact set. A point $s \in C$ is a *star point* if $[s,p] \subseteq C$ for all $p \in C$. Show that the set $\operatorname{St}(C)$ of star points of C is convex.

(10 points)

Exercise 5. Let $T = \text{conv}\{v_1, v_2, \ldots, v_{k+1}\}$ be a k-simplex. Show that $p \in \text{relint}(T)$ if and only if

$$p = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{k+1} v_{k+1} \qquad 1 = \lambda_1 + \lambda_2 + \cdots + \lambda_{k+1}$$

and $\lambda_i > 0$ for all $i = 1, 2, \ldots, k+1$.

(10 points)

Supplementary problems

Exercise 6. Prove the Gauss-Lucas theorem: If $p(z) \in \mathbb{C}[z]$ is a univariate polynomial with roots $r_1, \ldots, r_d$ then the roots of $p'(z)$ are contained in conv$\{r_1, \ldots, r_d\}$.
[Hint: If $p(z) = (z - r_1)(z - r_2) \cdots (z - r_d)$, then

$$p'(z) \;=\; \sum_j \prod_{i \neq j} (z - r_i).$$

For $s \in \mathbb{C}$ a root of $p'(z)$ expand $\overline{p(s)}p'(s)$ and use this to write s as a convex combination of the r_i.]