Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Arnau Padrol

SS2015

Discrete Geometry II

Bonus Homework — due July 21st

Bonus Exercise 1. i) Let $z_1, z_2, \ldots, z_n \in \mathbb{R}^d \setminus \{0\}$ and define the zonotope

$$Z \ := \ [0, z_1] + [0, z_2] + \cdots + [0, z_n]$$

Show that

$$V_d(Z) \ = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} |\det(z_{i_1}, z_{i_2}, \ldots, z_{i_d})|$$

ii) Consider the function $f(u) := V_{d-1}(\pi_u(Z))$ where $\pi_u : \mathbb{R}^d \to u^\perp \cong \mathbb{R}^{d-1}$ is the orthogonal projection onto $u^\perp = \{x \in \mathbb{R}^d : u^t x = 0\}$, the hyperplane perpendicular to u. Show that there is a zonotope $\Pi_Z = \sum_{j=1}^M [-w_j, w_j]$ such that

$$f(u) \ = \ \max\{u^t x : x \in \Pi_Z\}.$$

(10 points)

Bonus Exercise 2. Let $A \in \mathbb{R}^{n \times d}$ be a matrix of rank $d < n$ with positively dependent rows (i.e., $\alpha^t A = 0$ for some $\alpha \in \mathbb{R}^n_{>0}$). Let $\bar{A} = (\bar{a}_1, \ldots, \bar{a}_n) \in \mathbb{R}^{(n-d) \times n}$ be of rank $n - d$ and $\bar{A}A = 0$. For $b \in \mathbb{R}^n$ set $\bar{b} := \pi(b) = \bar{A}b$.

i) Show that $P_A(b) = \{x \in \mathbb{R}^d : Ax \le b\}$ is affinely isomorphic to $\{y \in \mathbb{R}^n : y \ge 0, \bar{A}y = \bar{b}\}$.

Now let

$$A \ = \ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

ii) Compute $\pi(\mathcal{B}_A)$ and the closed inner region $\bar{\mathcal{B}}_A$. These are two pointed 3-dimensional cones. Draw a corresponding 2-dimensional picture.

[Hint: This are questions about a planar point configuration.]

iii) (Bonus) How many normally non-equivalent types of simple polytopes with 6 facets are there for A?

(10+3 points)