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SS2015

Discrete Geometry II

Homework # 9 — due June 24th

Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **three** of your solutions. Only these will be graded. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. Let $e_1, \dots, e_d$ be the standard basis of $\mathbb{R}^d$, and set $e_0 := 0$. For $I \subseteq \{0, \dots, d\}$, set $\Delta_I = \text{conv}\{e_i \ : \ i \in I\}$.

i) Let $I_1, \dots, I_d \subseteq \{0, \dots, d\}$ of size two. Consider the graph $G(I_1, \dots, I_d)$ on the vertex set $\{0, \dots, d\}$ and with the edge $(i, j) \in G$ if and only if $\{i, j\} = I_k$ for some k.
Prove that $d! \operatorname{MV}(\Delta_{I_1}, \dots, \Delta_{I_d}) = 1$ if $G(I_1, \dots, I_d)$ is a tree and $\operatorname{MV}(\Delta_{I_1}, \dots, \Delta_{I_d}) = 0$ otherwise.

ii) For general $I_1, \dots, I_d \subset \{0, \dots, d\}$, prove that $\operatorname{MV}(\Delta_{I_1}, \dots, \Delta_{I_d}) \in \{\frac{1}{d!}, 0\}$. [Hint: Use monotonicity of the mixed volume.]

iii) Let G be a graph on the vertex set $\{0, \dots, d\}$ with edge set E. Prove that

$$V_d\left(\sum_{(i,j)\in E} \Delta_{\{i,j\}}\right) = \#\text{ spanning trees of } G$$

iv) [Bonus:] Compute the volume of the permutahedron.

(10+5 points)

Exercise 2. Let $P_1, \dots, P_d \subset \mathbb{R}^d$ be polytopes such that $0 \in \operatorname{relint}(P_i)$ for all i and $P = P_1 + \cdots + P_d$ is of dimension d. Show that the following statements are equivalent.

i) There is an exact mixed subdivision of P with a cell F of type $d(F) = (1, 1, \dots, 1)$.

ii) There are $u_i \in P_i$ such that $u_1, \dots, u_d$ are linearly independent.

iii) For every $I \subseteq [d]$

$$\dim \sum_{i \in I} P_i \ \geq \ |I|.$$

(10 points)

Exercise 3. i) Let $\Delta \subset \mathbb{R}^d$ be a d-simplex and let $P \subset \mathbb{R}^d$ be the result of truncating a vertex of Δ. Show that

$$\operatorname{MV}(\Delta[d-i], P[i]) \ = \ V(\Delta)$$

for all $d > i \geq 0$. Infer that the mixed volume is not *strictly* monotone with respect to inclusion.

ii) Let $K, L \subset \mathbb{R}^d$ be convex bodies and let $d \geq r > \dim K$. Show that

$$\mathrm{MV}(K[r], L[d-r]) \;=\; 0.$$

(10 points)

Exercise 4. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. For $a \in S^{d-1}$, we write π_a for the orthogonal projection onto $a^\perp$.

i) Prove that there is a polytope $\Pi(P) \subset \mathbb{R}^d$ such that

$$h_{\Pi(P)}(a) \;=\; V_{d-1}(\pi_a(P)).$$

ii) Prove that

$$\int_{S^{d-1}} V_{d-1}(\pi_a(P))\,\mathrm{d}a \;=\; C_d \cdot S(P),$$

where C_d is a constant independent of P.
[Hint: This can be done with or without i).]

iii) Given a polytope Q, can you give necessary conditions when $\Pi(P) = Q$ for some P? Is there a polytope P_d such that $\Pi(P_d) = C_d^\triangle$ where $C_d^\triangle$ is the d-dimensional cross-polytope?

iv) Bonus: Is there polytope P_d such that $\Pi(P_d) = \Pi_d$ where Π_d is the d-dimensional permutahedron?

v) Bonus: Let $\pi^a(x) := a^t x$ be the orthogonal projection onto the line $\mathbb{R}a$. Can you show that

$$\int_{S^{d-1}} V_1(\pi^a(P))\,\mathrm{d}a \;=\; c_d \cdot \mathrm{MV}(P, B_d, \ldots, B_d)$$

for some constant c_d that is independent of P?

(10+5+5 points)

Exercise 5. i) Prove the *weighted* arithmetic-geometric-mean inequality

$$\sum_{i=1}^n \lambda_i z_i \;\geq\; \prod_{i=1}^n z_i^{\lambda_i}$$

for all $z_i \geq 0$ and $\lambda_i \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$.

ii) Prove the *multiplicative* Brunn-Minkowski inequality:

$$V_d((1-\lambda)K + \lambda L) \;\geq\; V_d(K)^{1-\lambda} \cdot V_d(L)^\lambda$$

for all convex bodies $K, L \subset \mathbb{R}^d$ and $\lambda \in [0,1]$.

iii) What are the equality cases?

(10 points)