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Discrete Geometry II

Homework # 8 — due June 10th

Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. i) Let $P_1, \dots, P_n \subset \mathbb{R}^d$ be nonempty polytopes such that $P = P_1 + \cdots + P_n$ is full-dimensional and let $a \in \mathbb{R}^d$. Prove that P^a is a facet if and only if

$$(\lambda_1 P_1 + \cdots + \lambda_n P_n)^a$$

is a facet for any $\lambda_1, \dots, \lambda_n > 0$.

ii) Show that there is $A \in \mathbb{R}^{m \times d}$ and $b^1, \dots, b^n \in \mathbb{R}^m$ such that

$$\lambda_1 P_1 + \cdots + \lambda_n P_n = \{x \in \mathbb{R}^d : Ax \le \lambda_1 b^1 + \cdots + \lambda_n b^n\}$$

for all $\lambda_1, \dots, \lambda_n > 0$.

iii) [Bonus:] Let the rows of A be $a_1, \dots, a_m$. Use the formula

$$V_d(P) \;=\; \frac{1}{d} \sum_{i=1}^{m} h_P(a_i) V_{d-1}(P^{a_i})$$

to prove by induction on the dimension that $V_d(\lambda_1 P_1 + \cdots + \lambda_n P_n)$ is a homogeneous polynomial of degree d.

(10+3 points)

Exercise 2. For $i = 1, 2$, let V_i be finite subsets of $\mathbb{R}^d$ and $P_i = \text{conv}(V_i)$. Pick heights $\omega_i : V_i \to \mathbb{R}$ and consider the epigraphs

$$\mathcal{E}_i \;:=\; \text{conv}\{(v, \omega_i(v)) \;:\; v \in V_i\} + \{(0, t) : t \ge 0\}.$$

for $i = 1, 2$.

i) Prove that the bounded faces of $\mathcal{E}_1 + \mathcal{E}_2$ induce a mixed subdivision $\mathcal{M}$ of $P_1 + P_2$.

ii) Consider the regular subdivision $\mathcal{S}$ of $\mathrm{Cay}(P_1, P_2)$ induced by the heights ω_i. That is, the one corresponding to the bounded faces of the epigraph

$$\mathcal{F} \;:=\; \text{conv}\{(v, e_i, \omega_i(v)) \in \mathbb{R}^{d+2+1} \;:\; v \in V_i, i = 1, 2\} + \{(0, 0, t) : t \ge 0\}.$$

Prove that $\mathcal{M}$ is the mixed subdivision of $P_1 + P_2$ corresponding to $\mathcal{S}$.

(10 points)

Exercise 3. i) For a vector $u \in \mathbb{R}^d \setminus \{0\}$ let us write $[0, u] = \{\lambda u : 0 \le \lambda \le 1\}$. For a convex body $K \subset \mathbb{R}^d$, show that

$$V_d(\lambda K + \mu[0,u]) \;=\; \lambda^d V_d(K) + \lambda^{d-1}\mu\|u\| V_{d-1}(\pi_u(K))$$

where π_u is the orthogonal projection onto $\{x \in \mathbb{R}^d : u^t x = 0\}$, the hyperplane perpendicular to u.

ii) Let $z_1, \ldots, z_n \in \mathbb{R}^d$. Give an explicit description of

$$V_d(\lambda_1[0,z_1] + \cdots + \lambda_n[0,z_n])$$

for $\lambda_1, \ldots, \lambda_n \geq 0$.
[Hint: Consider $n = d$ first and think about what happens if $\lambda_i = 0$ for some i.]

(10 points)

Exercise 4. Let $S_i = [0,1] \subset \mathbb{R}$ for $i = 1, \ldots, n$. For $\omega_1, \ldots, \omega_n$, define

$$\mathcal{E}_i := [(0,0),(1,\omega_i)] + \{(0,t) : t \geq 0\}$$

Show that the fine mixed subdivisions of $[0,n]$ induced by $\mathcal{E}_1 + \cdots + \mathcal{E}_n$ are in bijection to permutations on $\{1, \ldots, n\}$.

[Bonus: Prove that the mixed subdivisions partially ordered by refinement are in bijection to the faces of the permutahedron Π_{n-1}.]

(10+3 points)

Bonus Exercise. Prove that if $\mathcal{S}$ is a mixed subdivision of $P_1 + \cdots + P_n$, then $\lambda\mathcal{S}$ is a mixed subdivision of $\lambda_1 P_1 + \cdots \lambda_n P_n$, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\lambda_i > 0$.

(+ 5 points)