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SS2015

Discrete Geometry II

Homework # 7 — due June 3rd

Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. Let $\mathcal{E} = \mathcal{E}(A, 0)$ for some positive definite matrix A.

i) Show that the polar of $\mathcal{E}$ is also an ellipsoid. That is, show that there is a positive definite matrix A' such that $\mathcal{E}^{\triangle} = \mathcal{E}(A', 0)$.

ii) Let $K \subset B_d$ be a d-dimensional convex body with $-K = K$. Show that the set $\{A \in \mathrm{PSD}_d : K \subseteq \mathcal{E}(A, 0), V(\mathcal{E}(A, 0)) \leq V(B_d)\}$ is closed and bounded.

iii) Infer that there exists an ellipsoid of minimum volume containing K and an ellipsoid of maximum volume contained in K.

(10 points)

Exercise 2. Let $A = \{p_1, \ldots, p_n\}$ be a configuration of points in the plane. Consider all the polygons conv$\{a_i \ : \ i \in I\}$ with the property that there is a a circle passing through the points $\{a_i \ : \ i \in I\}$ that leaves all the points $\{a_j \ : \ j \notin I\}$ outside.

i) Prove that these polygons form a regular subdivision of A. It is called the **Delaunay subdivision** of A.

ii) [Bonus:] Prove that the higher-dimensional analogue of the **Delaunay subdivision** is also regular (replace circle by sphere in the definition).

iii) [Bonus:] Find the relation between d-dimensional Delaunay subdivisions and $d+1$-dimensional **inscribed polytopes** (those that have all the vertices on a sphere).

(10+3+3 points)

Exercise 3. Let $A = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d$ be a configuration of points. For each $p_i \in A$, consider the region $V_i = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, p_i) \leq \mathrm{dist}(x, p_j) \forall j \in [n]\}$.

i) Prove that each V_i is a polyhedron and that the set of V_i's forms a polyhedral subdivision of $\mathbb{R}^d$. It is known as the **Voronoi diagram** of A.

ii) Prove that V_i is unbounded if and only if p_i lies in the boundary of conv(A).

iii) Define $\bar{V}_i = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, p_i) \geq \mathrm{dist}(x, p_j) \forall j \in [n]\}$. Prove that these $\bar{V}_i$ are polyhedra that are either empty or unbounded. The subdivision they form is known as the **farthest-point Voronoi diagram** of A.

(10 points)

Exercise 4. Let $P = \text{conv}\{v_0, v_1, \dots, v_d\}$ be a d-simplex. Consider the matrix

$$D = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & \ell_{00} & \ell_{01} & \cdots & \ell_{0d} \\ 1 & \ell_{10} & \ddots & & \ell_{1d} \\ \vdots & \vdots & & & \vdots \\ 1 & \ell_{d0} & \ell_{d1} & \cdots & \ell_{dd} \end{pmatrix} \qquad \text{where } \ell_{ij} = \|v_i - v_j\|^2.$$

Show that $2^d (d!)^2 V_d(P)^2 = (-1)^{d-1} \det(D)$.

(10 points)