Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Arnau Padrol

SS2015

# Discrete Geometry II

## Homework # 5 — due May 20th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a 🗊 are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

**Exercise 1.** i) Let $P$ be a $d$-polytope with facets $F_1, F_2, \ldots, F_m$. Show that

$$V_{d-1}(F_1) \; < \; \sum_{i=2}^{m} V_{d-1}(F_i)$$

ii) Prove Minkowski's theorem in the plane: If $u_1, \ldots, u_m \in \mathbb{R}^2$ are unit vectors spanning the plane and $\alpha_1, \ldots, \alpha_m > 0$ such that $\alpha_1 u_1 + \cdots + \alpha_m u_m = 0$, then there is a polygon $P$, unique up to translation, with outer facet normals $u_i$ and corresponding facet volumes $\alpha_i$.

**(10 points)**

**Exercise 2.** For a triangle $\Delta = \text{conv}\{a, b, c\}$ with ordered vertices $a, b, c \in \mathbb{R}^2$, let us define the **signed volume** of $\Delta$ as

$$V^\circ(\Delta) \; = \; \frac{1}{2} \det \begin{pmatrix} 1 & 1 & 1 \\ a & b & c \end{pmatrix}.$$

Let $P = \text{conv}\{v_1, \ldots, v_n\}$ be a polygon with vertices ordered counterclockwise, and let $p \in \text{int}(P)$ be a point in the interior.

i) Show that

$$(\star) \qquad V_2(P) \; = \; \frac{1}{2} \sum_{i=1}^{n} \det \begin{pmatrix} 1 & 1 & 1 \\ p & v_i & v_{i+1} \end{pmatrix} \qquad \text{where } v_{n+1} := v_1.$$

ii) Argue geometrically that the right-hand side of $(\star)$ is independent of the choice of $p \in \mathbb{R}^2$ (whether inside $P$ or not). This shows that the volume is a polynomial in the vertex coordinates, invariant under translation.

**(10 points)**