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SS2015

Discrete Geometry II

Homework # 4 — due May 13th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a ⧉ are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. i) Prove that if $A, B \in \mathcal{B}_d$ are polyboxes, then there are boxes $A_1, \dots, A_k$, $B_1, \dots, B_l, C_1, \dots, C_m$ such that

$$\begin{aligned} A &= A_1 \cup \cdots A_k \cup C_1 \cup \cdots \cup C_m \\ B &= B_1 \cup \cdots B_l \cup C_1 \cup \cdots \cup C_m \\ A \cup B &= A_1 \cup \cdots A_k \cup B_1 \cup \cdots B_l \cup C_1 \cup \cdots \cup C_m \\ A \cap B &= C_1 \cup \cdots \cup C_m \end{aligned}$$

are representations of polyboxes.

ii) Prove that the volume of a polybox is well-defined: For a polybox $S \in \mathcal{B}_d$, we can set

$$V(S) \ := \ V(B_1) + \cdots + V(B_k)$$

for any collection $B_1, \dots, B_k$ of boxes such that $S = B_1 \cup B_2 \cup \cdots \cup B_k$ and $\operatorname{int}(B_i) \cap \operatorname{int}(B_j) = \emptyset$ for all $i \neq j$.

(10 points)

Exercise 2. Let $P, Q \subset \mathbb{R}^d$ be d-simplices such that $P \cup Q$ is a simplex.

i) Prove that $\dim P \cap Q \geq d - 1$.

ii) Prove that if $\dim P \cap Q = d - 1$, then $P \cap Q$ is a facet of both.

iii) Assume that $P \cap Q$ is full-dimensional. Prove that a hyperplane H is facet-defining for $P \cap Q$ and $P \cup Q$ if and only if H is facet-defining for P and Q.

iv) Prove that $P \cap Q$ is also a simplex.
[Hint: Consider a representation of $P \cap Q$ in terms of inequalities.]

(10 points)

Exercise 3. i) Prove that any triangle is scissor-congruent to the square of the same area.
[Hint: go through a parallelogram]

ii) Prove that any two polygons of the same area are scissor-congruent.

(10 points)

Exercise 4. Let K, L be convex bodies in $\mathbb{R}^d$ such that $0 \in \text{int}(K)$ and $0 \in \text{int}(L)$.

i) Prove or disprove: For any $\epsilon > 0$ there exists $\delta > 0$ such that $d(K, L) < \delta$ implies $d(K^{\triangle}, L^{\triangle}) < \epsilon$.

ii) Show that if $|\frac{1}{d_K}(x) - \frac{1}{d_L}(x)| < \epsilon$ for all $x \in \mathbb{S}^{d-1}$, then $d(K, L) < \epsilon$.

iii) Assume additionally that both K and L are contained in $B_R(0)$. Show that then $|d_K(x) - d_L(x)| < \frac{\epsilon}{R^2}$ for all $x \in \mathbb{S}^{d-1}$ implies $d(K, L) < \epsilon$.

(10 points)