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SS2015

Discrete Geometry II

Homework # 3 — due May 6th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a ⧉ are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. Let $K \subset \mathbb{R}^d$ be a convex body with $0 \in \text{int}(K)$.

i) For an (invertible) matrix $U \in \mathbb{R}^{d \times d}$, we write $UK = \{Ux : x \in K\}$. Show that

$$(UK)^{\triangle} \; = \; (U^{-1})^t K^{\triangle}$$

ii) If $K \subset \mathbb{R}^d$ is the unit ball centered at the origin, then $K = K^{\triangle}$.

iii) Show that if $K = K^{\triangle}$, then $K = B_d$ is the unit ball.

iv) (Bonus) Let $v \in \mathbb{R}^d$ such that $0 \in \text{int}(v + K)$. Prove that

$$(v+K)^{\triangle} \; = \; \left\{ \tfrac{1}{1+v^t x} x \; : \; x \in K^{\triangle} \right\}.$$

(10+3 points)

Exercise 2. Consider the following polytope in $\mathbb{R}^4$:

$$P := \text{conv}\{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\},$$

where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$.

i) Show that $P^{\triangle} \supseteq Q$ and $P \subseteq Q^{\triangle}$, where Q is the convex hull of the 24 points

$$\pm e_1, \pm e_2, \pm e_3, \pm e_4, \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$$

ii) Show that $P^{\triangle} = Q$.

iii) (Bonus) Show that $P^{\triangle}$ is congruent to a scaled copy of P.

The polytope P is called 24-cell.

(10+3 points)

Exercise 3. A closed convex cone $C \subset \mathbb{R}^d$ is **self-dual** if $C^{\triangle} = -C$.

i) Show that $\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0\}$ is self-dual.

ii) Let

$$\mathcal{L}_d \; := \; \{(t,x) \in \mathbb{R}^{d+1} : t^2 - x_1^2 - \cdots - x_d^2 \geq 0, t \geq 0\}$$

be the **Lorentz cone**. Prove that $\mathcal{L}_d$ is a self-dual closed convex cone.

iii) (Bonus) Show that PSD_d is a self-dual cone.
[Hint: The *right* inner product on matrices is $\langle A, B \rangle = \mathrm{tr}(A^t B) = \sum_{ij} A_{ij} B_{ij}$]

(10+3 points)

Exercise 4. Let $K \subset \mathbb{R}^d$ be a convex body. The **diameter** of K is the maximal distance between a pair of points of K:

$$\operatorname{diam}(K) = \max_{x,y \in K} \operatorname{dist}(x, y).$$

i) Let $p, q \in K$ such that $\operatorname{dist}(p, q) = \operatorname{diam}(K)$. Prove that p and q are exposed points of K.

ii) Deduce that any convex body containing more than one point has at least two exposed points. Show that this is not necessarily true if the set is not compact.

(10 points)