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Discrete Geometry II

Homework # 2 — due April 29th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the lecture on Wednesday. Please write different exercises in different sheets.

Exercise 1. Let $\mathbb{R}^{d\times d}_{\text{sym}}$ be the vector space of symmetric d-by-d matrices and let $\mathrm{PSD}_d \subseteq \mathbb{R}^{d\times d}_{\text{sym}}$ be the closed convex cone of positive semi-definite matrices.

i) Show that PSD_d is full-dimensional.

ii) For $u \in \mathbb{R}^d \setminus \{0\}$ define the symmetric matrix $u \bullet u \in \mathbb{R}^{d\times d}_{\text{sym}}$ by

$$(u \bullet u)_{ij} := u_i u_j \quad \text{for } 1 \leq i, j \leq d.$$

Show that $u \bullet u$ is positive semi-definite of rank 1.

iii) Let $S = \{u \bullet u : u \neq 0\}$. Show that for every $A \in \mathrm{PSD}_d$ there $U_1, \dots, U_k \in S$ with $k \leq d$ such that $A = \sum_i U_i$.
Deduce that $\mathrm{PSD}_d = \mathrm{conv}(S \cup 0)$.
[Hint: You may want to brush up on your linear algebra. For example, the fact that symmetric matrices are orthogonally diagonalizable.]

iv) (Bonus) For a convex cone C an **extreme ray** is a vector u such that $C \setminus \mathbb{R}_{\geq 0} u$ is convex. Show that S is the set of extreme rays of PSD_d.
[Hint: Argue about the rank of matrices in PSD_d]

(10+3 points)

Exercise 2. Let $K_1, K_2, L \subset \mathbb{R}^d$ be convex bodies.

i) Show that $h_{K_1+L}(c) = h_{K_1}(c) + h_L(c)$ for all $c \in \mathbb{R}^d \setminus \{0\}$.
What is the support function of $K_1 + L$ if $L = \{p\}$ is a point?

ii) Show that if $K_1 \cup K_2$ is convex, then $h_{K_1 \cap K_2}(c) = \min(h_{K_1}(c), h_{K_2}(c))$.
Show that this is no longer true if $K_1 \cup K_2$ is not convex.

iii) Show that if $K_1 \cup K_2$ is convex, then

$$(K_1 \cup K_2) + (K_1 \cap K_2) \;=\; K_1 + K_2.$$

(10 points)

Exercise 3. i) Let $A, B \subset \mathbb{R}^d$ be convex sets (not necessarily closed or bounded). Show that A and B can be separated if and only if $A + (-B)$ can be separated from the origin 0.

ii) Find:

- a closed set $S \subset \mathbb{R}^d$ such that $\mathrm{conv}(S)$ is not closed.

- a linear projection of a closed convex set S that is not closed.
- two closed convex sets $K, L \subset \mathbb{R}^d$ such that $K + L$ is not closed.
- two closed convex sets $K, L \subset \mathbb{R}^d$ that are in disjoint open halfspaces but cannot be in two disjoint closed halfspaces.

(10 points)

Exercise 4. Prove that any two disjoint closed convex sets $K, L \subset \mathbb{R}^d$, one of which is compact, can be strictly separated by a hyperplane.

(10 points)