Discrete Geometry II
Discrete Convex Geometry
(i.e. this ○, not this )

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As of now, these are *just* my notes. They are not necessarily complete and they are probably crawling with mistakes! What counts is what is said during the lectures and the recitations. The first version of these notes were prepared by Miriam Schlöter during the course given in the summer term 2013. Thanks a lot!

Lecture 1, April 14

1. Basic convex geometry

1.1. Definitions and Examples. We recall the following notions from linear algebra. A set $U \subseteq \mathbb{R}^d$ is a **linear subspace** if $\lambda x + \mu y \in U$ for all $x, y \in U$ and $\lambda, \mu \in \mathbb{R}$. Any intersection of linear subspaces is a linear subspace. The **linear hull** $\mathrm{lin}(S)$ of a set $S \subseteq \mathbb{R}^d$ as the intersection of all linear subspaces U containing S. It is the unique, inclusion-minimal linear subspace containing S. Verify that

$$\mathrm{lin}(S) \;=\; \Big\{\lambda_1 s_1 + \cdots + \lambda_k s_k : k \geq 1, \lambda_1, \ldots, \lambda_k \in \mathbb{R}, s_1, \ldots, s_k \in S\Big\}.$$

The collection of elements $S = \{s_1, \ldots, s_n\}$ is **linearly independent** if $\mathrm{lin}(S) \neq \mathrm{lin}(S \setminus \{s_i\})$ for all i. That is,

$$\lambda_1 s_1 + \cdots + \lambda_n s_n = 0 \quad \Longrightarrow \quad \lambda_1 = \cdots = \lambda_n = 0.$$

A set $A \subseteq \mathbb{R}^d$ is an **affine subspace** if it is of the form $A = t + U$ for some $t \in \mathbb{R}^d$ and some linear subspace $U \subseteq \mathbb{R}^d$. Equivalently, for $a_1, \ldots, a_k \in A$ we have

$$\lambda_1 a_1 + \cdots + \lambda_k a_k \in A$$

for every $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. The **affine hull** $\mathrm{aff}(S)$ of a set S is the intersection of all affine subspaces containing S. For $S = \{s_1, \ldots, s_n\}$ the affine hull is given as

$$\mathrm{aff}(s_1, \ldots, s_n) \;=\; \{\lambda_1 s_1 + \cdots + \lambda_n s_n : \lambda_1 + \cdots + \lambda_n s_n\} \;=\; s_1 + \mathrm{lin}(s_2 - s_1, \ldots, s_n - s_1).$$

It follows that $S = \{s_1, \ldots, s_n\}$ is **affinely independent** if

eqn:aff_indep

$$\begin{array}{ll} \lambda_1 s_1 + \cdots + \lambda_n s_n &= 0 \\ \lambda_1 \quad\; + \cdots + \lambda_n &= 0 \end{array} \quad \Longrightarrow \quad \lambda_1 = \cdots = \lambda_n = 0. \tag{1}$$

In particular an **affine line** is the affine hull of two distinct points $a, b \in \mathbb{R}^d$

$$\overline{ab} \;:=\; \mathrm{aff}(a, b) = \{(1-\lambda)a + \lambda b : \lambda \in \mathbb{R}\}.$$

The (oriented) **line segment** $[a, b]$ between a and b is

$$[a, b] \;:=\; \{(1-\lambda)a + \lambda b : 0 \leq \lambda \leq 1\}$$

Definition 1 (Convex set, Convex bodies). A set $K \subseteq \mathbb{R}^d$ is called **convex** if $[a, b] \subseteq K$ for all $a, b \in K$. A compact convex set K is called a **convex body**.

Of course, linear and affine subspaces are convex. A convex set K is called **line-free** if $\overline{ab} \notin K$ for all $a, b \in K$ with $a \neq b$.

Here are some examples of convex sets.

ex:norms

Example 2 (Norms and balls). The (Euclidean) **unit ball** is

$$B_d = \Big\{x \in \mathbb{R}^d : \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2} \leq 1\Big\}.$$

We can show, that the unit ball is convex using the triangle inequality: Let $x, y \in B_d$ and $0 \leq \lambda \leq 1$

$$\|(1-\lambda)x + \lambda y\|_2 \;\leq\; (1-\lambda)\|x\|_2 + \lambda\|y\|_2 \;\leq\; 1.$$

This clearly holds for any norm! For example the cubes $C_d = [-1, +1]^d$ are the unit balls for the maximum norm $\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, d\}$ and the octahedron is the unit ball for the 1-norm $\|x\|_1 = |x_1| + \cdots + |x_d|$ for $d = 3$.

On the other hand we can also use convex bodies to construct norms: For a convex body $K \subset \mathbb{R}^d$, define $\|p\|_K$ for a point $p \in \mathbb{R}^d$ by

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}.$$

This satisfies the triangle inequality but not necessarily the symmetry $\|-p\|_K = \|p\|_K$. For that, we need to require that K is **centrally-symmetric**, that is, $K = -K$.

EXAMPLE 3 (PSD cone). A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called **positive semi-definite (psd)** if all eigenvalues are nonnegative. The set

$$\mathrm{PSD}_n := \{A \in \mathbb{R}^{n\times n} : A = A^t \text{ and } A \text{ is psd}\}$$

is called the **PSD cone**. It is a closed convex set. To see this, recall the well-known fact from linear algebra that A is positive semi-definite if $v^tAv \geq 0$ for all $v \in \mathbb{R}^n$. For fixed $v \in \mathbb{R}^d$ the expression v^tAv is linear in A. Hence, for $A, B \in \mathrm{PSD}_n$ and $0 \leq \lambda \leq 1$ we get

$$v^t((1-\lambda)A + \lambda B)v = (1-\lambda)\underbrace{v^tAv}_{\geq 0} + \lambda\underbrace{v^tBv}_{\geq 0} \;\geq\; 0,$$

which implies $(1-\lambda)A + \lambda B \in \mathrm{PSD}_n$. In fact, PSD_n is a **convex cone**, that is, $\lambda A + \mu B \in \mathrm{PSD}_n$ for all $A, B \in \mathrm{PSD}_n$ and $\lambda, \mu \geq 0$.

Lecture 2, April 15

EXAMPLE 4 (Positive polynomials and sums-of-squares). Among the set of polynomials $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ in n variables of degree $2k$, the **positive polynomials**

$$\mathcal{P}_{d,2k} \;:=\; \{p \in \mathbb{R}[\mathbf{x}] : \deg(p) \leq 2k, p(q) \geq 0 \text{ for all } q \in \mathbb{R}^n\}$$

and the set of **sums-of-squares**

$$\Sigma_{d,2k} \;:=\; \{p(x) = h_1(x)^2 + \cdots + h_n(x)^2 : h_1, \dots, h_n \in \mathbb{R}[\mathbf{x}], \deg(h_i) \leq k\}$$

are closed convex cones.
We have $\Sigma_{d,2k} \subseteq \mathcal{P}_{d,2k}$ and but in general $\Sigma_{d,2k} \neq \mathcal{P}_{d,2k}$ (Hilbert). Interestingly, it can be (easily) shown that $\Sigma_{d,2k} = \mathrm{PSD}_N$ for $N = \binom{n+d-1}{d}$.

EXAMPLE 5 (Convex functions). A function $f : \mathbb{R}^d \to \mathbb{R}$ is called **convex** if for all $x, y \in \mathbb{R}^d$ and $0 \leq \lambda \leq 1$

$$f((1-\lambda)x + \lambda y) \;\leq\; (1-\lambda)f(x) + \lambda f(y).$$

for all $x, y \in \mathbb{R}^d$ and $0 \leq \lambda \leq 1$. The **epigraph** of a function f is defined as

$$\mathrm{epi}(f) \;:=\; \{(x, t) \in \mathbb{R}^d \times \mathbb{R} : t \geq f(x)\}.$$

is a convex set if and only if f is convex. In this case, $\{x \in \mathbb{R}^d : f(x) \leq c\}$ is convex for any $c \in \mathbb{R}$.

On the other hand, we can view convex functions as a subset of the vector space of functions on $\mathbb{R}^d$. In fact, we can restrict to continuous functions.

PROPOSITION 6. If $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then f is continuous.

Let us write $C^0(\mathbb{R}^d)$ for the $\mathbb{R}$-vector space of real-valued continuous functions on $\mathbb{R}^d$. From the definition of convex function it follows that

$$C_{\mathrm{conv}}(\mathbb{R}^d) \;:=\; \{f : \mathbb{R}^d \to \mathbb{R} : f \text{ convex}\} \subset C^0(\mathbb{R}^d)$$

is a convex cone.
Of course, convex (regular) polygons and polyhedra such as the platonic solids (tetrahedra, cube, octahedron, dodecahedron, icosahedron) are convex bodies but to describe these, we need tools.

1.2. Convex hulls and Carathéodory numbers. The intersection of an arbitrary collection of convex sets is convex. This prompts the following definition.

DEFINITION 7 (Convex hull). Let $S \subset \mathbb{R}^d$. The **convex hull** of S is the convex set

$$\mathrm{conv}(S) = \bigcap\{K \subseteq \mathbb{R}^d \text{ convex } : S \subseteq K\}.$$

By definition, $\mathrm{conv}(S)$ is the inclusion-minimal convex set containing S. This is nice but hard to work with. The following gives a description of the convex hull in terms of the points of S.

THEOREM 8. *For $S \subseteq \mathbb{R}^d$ the convex hull is given by*

$$\mathrm{conv}(S) \;=\; \Big\{\lambda_1 s_1 + \cdots + \lambda_k s_k : k \geq 1, s_1, \dots, s_k \in S, \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1, \lambda_1, \dots, \lambda_k \geq 0\Big\}.$$

Proof. We prove the two inclusions:

$\subseteq$: It suffices to show that the right-hand side is a convex set. Since it clearly contains S, the inclusion follows from the definition of convex hulls. Let $a = \alpha_1 s_1 + \cdots + \alpha_k s_k$ for $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$ and $b = \beta_1 s_1 + \cdots + \beta_k s_k$ for $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. For $0 \leq \lambda \leq 1$, we compute

$$(1-\lambda)a + \lambda b \;=\; \sum_i \underbrace{\left((1-\lambda)\alpha_i + \lambda\beta_i\right)}_{=:\gamma_i} s_i$$

Since all the involved scalars are nonnegative, see that $\gamma_i \geq 0$ and

$$\sum_i \gamma_i \;=\; (1-\lambda)\sum_i \alpha_i + \lambda \sum_i \beta_i \;=\; (1-\lambda)1 + \lambda 1 \;=\; 1.$$

$\supseteq$: For this inclusion, we show if K is a convex set containing S, then K has to contain the right-hand side. For that we have to show that for all $k \geq 1$ and $s_1, \dots, s_k \in S$, $\lambda_1, \dots, \lambda_k \geq 0$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$ we have

eqn:conv_hull

$$\lambda_1 s_1 + \cdots + \lambda_k s_k \in K. \tag{2}$$

We prove this by induction on k: For $k = 1$ this asserts that $S \subseteq K$. For $k = 2$, this states that $[s_1, s_2] \subseteq K$ for any $s_1, s_2 \in S$ which is true since K is convex. Let us assume that $k > 2$ and that (2) holds for any collection of $\leq k-1$ points. Now, for $s_1, \dots, s_k \in S$ and $\lambda_1, \dots, \lambda_k \geq 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. If any $\lambda_j = 0$, then this is a convex combination of $\leq k-1$ points and we are done. Hence, $\lambda_j > 0$ for all j and therefore $\lambda_j < 1$ for all j. We compute

$$p \;=\; \lambda_1 s_1 + \cdots + \lambda_k s_k \;=\; (1-\lambda_k)\underbrace{\left[\frac{1}{1-\lambda_k}(\lambda_1 s_1 + \cdots + \lambda_{k-1}s_{k-1})\right]}_{\in K \text{ by induction}} + \lambda_k s_k.$$

Since K is convex, we infer that $p \in K$. $\square$

In light of this result, we call

$$p \;=\; \lambda_1 s_1 + \cdots + \lambda_n s_n, \quad \lambda_1, \dots, \lambda_n \;\geq\; 0, \quad \lambda_1 + \cdots + \lambda_n \;=\; 1$$

a **convex combination** of $s_1, \dots, s_n$.

We digress to give a short but beautiful application of convex hulls to complex polynomials: Let $p(t) \in \mathbb{R}[t]$ be a polynomial with only real roots. By the intermediate value theorem we know that the roots of the derivative $p'(t)$ are interlaced with the roots of $p(t)$ and therefore in the interval between the smallest and the largest root of $p(t)$. The Gauß-Lucas Theorem generalizes this to the complex case.

thm:conv

Theorem 9. *Let $p(t) \in \mathbb{C}[t]$ be a univariate polynomial with roots $r_1, \dots, r_k \in \mathbb{C}$. Then the roots of the derivative $p'(t)$ are contained in* $\mathrm{conv}(r_1, \dots, r_k) \subseteq \mathbb{C} \cong \mathbb{R}^2$.

Proof. Exercise. $\square$

A convex set P is a **polytope** if $P = \mathrm{conv}(S)$ for some finite set S. The $(n-1)$-dimensional **standard simplex** Δ_{n-1} or (**$(n-1)$-simplex** for short) is the polytope

$$\Delta_{n-1} \;:=\; \{\lambda \in \mathbb{R}^n : \lambda_1, \dots, \lambda_n \geq 0, \lambda_1 + \cdots + \lambda_n = 1\} \;=\; \mathrm{conv}(e_1, \dots, e_n).$$

polytope_simplex

Proposition 10. Every polytope $P = \mathrm{conv}(p_1, \dots, p_n) \subset \mathbb{R}^d$ is the linear projection of the $(n-1)$-simplex. Moreover, for any S and point $p \in \mathrm{conv}(S)$ there is a polytope $P \subset \mathrm{conv}(S)$ with $p \in P$.

Proof. Consider the linear map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ defined on the standard basis by $\pi(e_i) = p_i$ for $1 \leq i \leq n$. For $p \in P$ there are, by definition, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and $p = \sum_i \lambda_i p_i$. Again by definition $\lambda = (\lambda_1, \dots, \lambda_n)$ is a point of Δ_{n-1} and we compute

$$\pi(\lambda) \;=\; \pi(\lambda_1 e_1 + \cdots + \lambda_n e_n) \;=\; \lambda_1 \pi(e_1) + \cdots + \lambda_n \pi(e_n) \;=\; \lambda_1 p_1 + \cdots + \lambda_n p_n \;=\; p.$$

For $p \in \mathrm{conv}(S)$ there are $s_1, \dots, s_k \in S$ such that $p \in \mathrm{conv}(s_1, \dots, s_k)$. $\square$

Proposition 10 makes one curious if there is a uniform bound on how many points are needed to represent $p \in \operatorname{conv}(S)$ as a convex combination of elements in S. A uniform bound is provided by the following theorem which is a basic but important tool in convex geometry.

hm:caratheodory

THEOREM 11 (Carathéodory's Theorem – preliminary version). *Let $K = \operatorname{conv}(S) \subseteq \mathbb{R}^d$ be a convex set. Then every $p \in K$ is in the convex hull of at most $d+1$ points of S.*

PROOF. For $p \in \operatorname{conv}(S)$, let $s_1, s_2, \dots, s_k \in S$ and $\lambda_1, \lambda_2, \dots, \lambda_k \geq 0$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and

$$p = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_k s_k.$$

Let us assume that k is the minimal number of elements of S necessary to represent p as a convex combination. In particular, this implies that $\lambda_i > 0$ for all $i = 1, \dots, k$. Arguing by contradiction, we assume that $k \geq d+2$. Consider the matrix

$$\begin{pmatrix} s_1 & s_2 & \cdots & s_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times n}.$$

Since $n > d+1$, there is an element in the kernel. That is, there are $\mu_1, \dots, \mu_n \in \mathbb{R}^n$ not all zero such that

$$\mu_1 s_1 + \cdots + \mu_n s_n = 0 \quad \text{and} \quad \mu_1 + \cdots + \mu_n = 0.$$

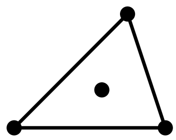
Thinking back to (1) this means $s_1, \dots, s_n$ is **affinely dependent**.

Let $\varepsilon = \min\{\frac{\lambda_i}{-\mu_i} : \mu_i < 0\}$ and define $\lambda_i' := \lambda_i + \varepsilon\mu_i$ for $i = 1, \dots, n$. You can check that

$$p = \lambda_1' s_1 + \cdots + \lambda_n' s_n, \quad \lambda_1', \dots, \lambda_n' \geq 0, \quad \lambda_1' + \cdots + \lambda_n' = 1.$$

Moreover, for an index j for which $\varepsilon = \frac{\lambda_j}{-\mu_j}$, we see that $\lambda_j' = 0$. Hence $p \in \operatorname{conv}(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_k)$ which contradicts our assumption that k is minimal. □

In the general case, the bound given in the theorem is sharp: Take $S = \{s_1, s_2, s_3\} \subset \mathbb{R}^2$ the vertices of a triangle. The points are affinely independent and every point of $K = \operatorname{conv}(s_1, s_2, s_3)$ as a unique expression as a convex combination. Thus for the midpoint $\frac{1}{3}(s_1 + s_2 + s_3)$ all three points are need.



Of course there are situations were fewer points than $d+1$ are needed. For example, if $S = \operatorname{conv}(S)$, then every point is represented by itself. We define the **Carathéodory number** $\mathcal{C}(S)$ of S as the minimal k such that every $p \in \operatorname{conv}(S)$ is a convex combination of at most k points in S. (We will make this intrinsic to the convex set $\operatorname{conv}(S)$ as soon as we defined extreme points.)

A nontrivial example where $\mathcal{C}(S) < d+1$ is the unit sphere $S = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$. Let $p \in B_d = \operatorname{conv}(S)$ be a point and let L be an affine line passing through p. Then L meets S in two points $p_1, p_2 \in S$ and $p \in [p_1, p_2] \subseteq S$.

For now we can use Carathéodory's theorem to prove a basic and somewhat intuitive fact. Recall that a set $S \subset \mathbb{R}^d$ is **compact** if every open cover has a finite subcover. In the Euclidean case this is equivalent to the requirement that S is closed and bounded. For example, the standard simplices Δ_n are compact.

or:compact_conv

COROLLARY 12. Let $S \subseteq \mathbb{R}^d$ be a compact set. Then $\operatorname{conv}(S)$ is compact.

PROOF. Define the map

$$\pi : \underbrace{S \times \cdots \times S}_{d+1} \times \Delta_d \longrightarrow \mathbb{R}^d, (s_1, \dots, s_{d+1}, \lambda_1, \dots, \lambda_{d+1}) \mapsto \lambda_1 s_1 + \cdots + \lambda_{d+1} s_{d+1}$$

By Carathéodory's theorem we have $\pi(S^{d+1} \times \Delta_d) = \operatorname{conv}(S)$. The set $S^{d+1} \times \Delta_d$ is compact and since π is continuous so is its image. □

The bound of Carathéodory's theorem can be improved by taking topological properties of S into account; we will see some of this later.

1.3. Topological Properties. We write $B_\varepsilon(p) := p + \varepsilon B_d$ for the ε-ball centered at p.

DEFINITION 13 (Interior and boundary). Let $X \subseteq \mathbb{R}^d$. A point $p \in X$ is an **interior point** if $B_\varepsilon(p) \subseteq X$ for some $\varepsilon > 0$. The **interior** $\operatorname{int}(X) \subseteq X$ is the set of interior points.
A point $p \in X$ is a **boundary point** if $B_\varepsilon(p) \nsubseteq X$ for all $\varepsilon > 0$.

Naturally, we have $X = \operatorname{int}(X) \cup \partial X$ and $\operatorname{int}(X) \cap \partial X = \varnothing$.

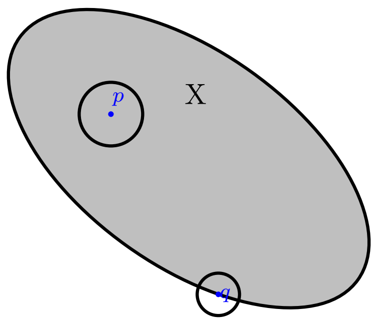


FIGURE 1. p interior point, q boundary point

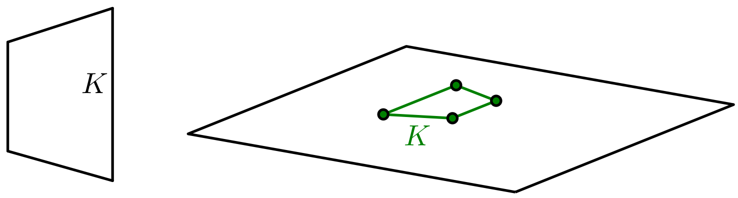


FIGURE 2. Left: K in $\mathbb{R}^2$ has interior, right: K in $\mathbb{R}^3$ has only boundary points

As convexity is intrinsic notion to a given set K, we want to talk about interior and boundary points independent of the embedding (see example below). By definition, the inclusion-minimal affine subspace containing K is the affine hull $\operatorname{aff}(K)$. We can use $\operatorname{aff}(K)$ as a canonical embedding of K into a Euclidean space and define the notion of interior and boundary relative to it.

DEFINITION 14 (Relative interior). Let $K \subset \mathbb{R}^n$ be a convex set and $p \in K$. Then p is in the **relative interior** of K if

$$B_\varepsilon(p) \cap \operatorname{aff}(K) \subseteq K$$

for some $\varepsilon > 0$ and we write $\operatorname{relint}(K)$ for the set of relative interior points.

The relative boundary is defined likewise. However, we will not make a distinction between the boundary and the relative boundary and simply state $\partial K = K \setminus \operatorname{relint}(K)$.

PROPOSITION 15. Let K be a convex set. Then $\operatorname{relint}(K)$ is convex.

PROOF. We may assume that $\operatorname{aff}(K) = \mathbb{R}^d$ and thus $\operatorname{relint}(K) = \operatorname{int}(K)$. Let $p_0, p_1 \in \operatorname{int}(K)$. This means there are $\varepsilon_0, \varepsilon_1 > 0$ such that $B_{\varepsilon_i}(p_i) \subseteq K$. We want to show that $[p_0, p_1] \subseteq \operatorname{int}(K)$. Fix $0 \le \lambda \le 1$ and set $p := (1-\lambda)p_0 + \lambda p_1$ and $\varepsilon := (1-\lambda)\varepsilon_0 + \lambda\varepsilon_1$. We claim that $B_\varepsilon(p) \subseteq \operatorname{int}(K)$. A point of $B_\varepsilon(p)$ is of the form $y = p + \varepsilon u$ where u satisfies $\|u\| \le 1$. Now define $y_i := p_i + \varepsilon_i u$ for $i = 0, 1$. By construction $y_i \in B_{\varepsilon_i}(p_i) \subseteq K$ and $y = (1-\lambda)y_0 + \lambda y_1$. By convexity of K it follows that $y \in K$ which shows that $B_\varepsilon(p) \subseteq K$. $\square$

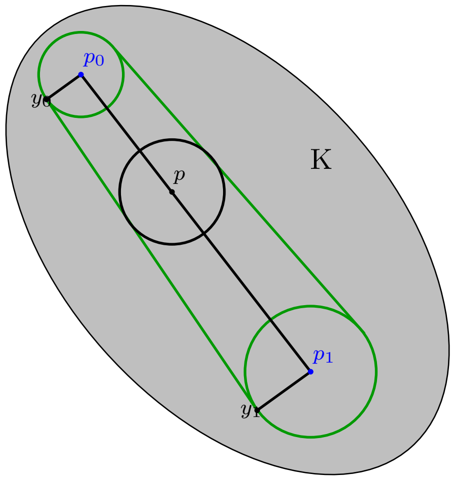


FIGURE 3.

The affine hull is the smallest affine subspace containing K in which K has an interior. This prompts the definition of dimension of K in terms of the dimension of its affine hull. The dimension of an affine subspace $A = t + U$ is defined as $\dim A = \dim U$. Equivalently, it is the maximal number of affinely independent elements of A minus 1.

DEFINITION 16 (Dimension). For a convex set $K \subseteq \mathbb{R}^d$ the **dimension** is

$$\dim(K) \;:=\; \dim \operatorname{aff}(K)$$

relint_nonempty COROLLARY 17. Let $K \neq \varnothing$ be a convex set. Then $\operatorname{relint}(K) \neq \varnothing$.

PROOF. Exercise! □

With the notion of dimension, we can free Carathéodory's theorem from the dependence of the ambient space.

THEOREM 18 (Carathéodory's Theorem – final version). *Let $K = \operatorname{conv}(S)$ be a convex set of dimension d. Then every $p \in K$ is in the convex hull of at most $d+1$ points of S.*

COROLLARY 19. Let $P = \operatorname{conv}(x_1, .., x_n)$. If $q \in \operatorname{relint} P$, then there are $\lambda_1, \ldots, \lambda_n > 0$ with $\lambda_1 + \cdots + \lambda_n = 1$ and

$$q \;=\; \lambda_1 x_1 + \cdots + \lambda_n x_n.$$

Please note that the statement is *not* that any representation of q as a convex combination has all coefficients strictly positive.

PROOF. We can assume that $P \subset \mathbb{R}^d$ is full-dimensional. Let $z = \frac{1}{n}\sum_{i=1}^n x_i$ be the **barycenter** of P. If $x = z$, then we are done.

Hence, we assume that $x \neq z$. Since q is an interior point, we can find a point $q_0 \in \overline{qz} \cap K$ such that $q = (1-\lambda)q_0 + \lambda z$ for some $0 < \lambda < 1$. Now $q \in P$ and hence $q = \sum_i \mu_i x_i$ for some coefficients $\mu_i \geq 0$ and $\sum_i \lambda_i = 1$. Setting $\lambda_i := (1-\lambda)\mu_i + \lambda\frac{1}{n}$ yields the claim. □

Lecture 3, April 21

We next turn to the question if for every convex body K there is some distinguished (minimal?) set S such that $K = \operatorname{conv}(S)$.

DEFINITION 20 (Extreme points). A point $v \in K$ is **extreme** if $v = (1-\lambda)x + \lambda y$ with $x, y \in K$ and $0 < \lambda < 1$ implies $x = y = v$. Equivalently $K \setminus \{v\}$ is convex. We define

$$\operatorname{ext}(K) \;:=\; \{v \in K : v \text{ is an extreme point}\}.$$

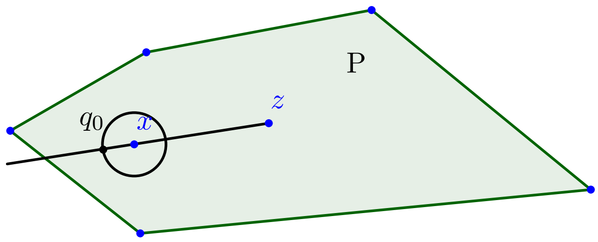


FIGURE 4.

Clearly, if $K = \mathrm{conv}(S)$ then $\mathrm{ext}(K) \subseteq S$.

conv_ext_points THEOREM 21 (Minkowski). *If K be a convex body, then $K = \mathrm{conv}(\mathrm{ext}(K))$.*

1.4. Support and Separation. Let $A \subseteq \mathbb{R}^d$ be a closed, convex set. For $x \in \mathbb{R}^d$ let $y \in A$ be a point such that $\mathrm{dist}(x, y) = \|x - y\| = m$ is minimal. To see that such a point exists, pick some $z \in A$ and let $r = \mathrm{dist}(x, z)$. Then $B_r(x) \cap A \neq \varnothing$ is compact, thus $\mathrm{dist}(x, \cdot)$ attains a minimum over $B_r(x) \cap A$ in some point y.

It is easy to see that for $x \in \mathbb{R}^d$, the point $y \in A$ is unique: Assume $y_0, y_1 \in A$ both have distance m from x. Then

$$\|\tfrac{1}{2}(y_0 + y_1) - x\| \;=\; \tfrac{1}{2}\|y_0 - x + y_1 - x\| \;\leq\; \tfrac{1}{2}(m + m)$$

and equality is attained if $y_0 = y_1$. Existence and uniqueness imply that there is a unique map $\pi_A : \mathbb{R}^d \to A$, called the **nearest point map**, such that $\mathrm{dist}(x, \pi_A(x))$ is minimal for all $x \in \mathbb{R}^d$.

The following theorem due to Motzkin (and which we will not prove) asserts that the uniqueness of π_A is characteristic for closed convex sets.

THEOREM 22. *Let A be a closed set such that the nearest point map exists and is unique. Then A is convex.*

So far we have been looking at convex sets from an *intrinsic* point of view. We will now see how they can be described from the outside. An **(oriented) affine hyperplane** is a set of the form

$$H \;:=\; \{x \in \mathbb{R}^d : c^t x = \delta\}$$

$c \in \mathbb{R}^d \setminus \{0\}$ and $\delta \in \mathbb{R}$. This is an affine subspace of dimension $d - 1$. We call H a **linear hyperplane** if $\delta = 0$. Each (oriented) affine hyperplane H induces two **halfspaces**

$$H^{\leq} \;=\; \{x \in \mathbb{R}^d : c^t x \leq \delta\}$$
$$H^{\geq} \;=\; \{x \in \mathbb{R}^d : c^t x \geq \delta\}$$

Verify that halfspaces are always closed and convex. We also define the corresponding **open halfspaces** $H^{>}$ and $H^{<}$.

DEFINITION 23 (Separating hyperplane). Let $A, B \subseteq \mathbb{R}^d$ be two sets. A hyperplane H is called a **(properly) separating hyperplane** if $A \subseteq H^{\leq}$, $B \subseteq H^{\geq}$ and $A \cup B \notin H$.

We say that H **strictly** separates A and B if $A \subseteq H^{<}$ and $B \subseteq H^{>}$.

Here is a first situation when separating hyperplanes exist.

thm:sep THEOREM 24 (Separation Theorem). *Let $A \subseteq \mathbb{R}^d$, $A \neq \varnothing$ be a closed, convex set and $p \notin A$. Then there is a hyperplane that strictly separates p from A.*

PROOF. Let $q = \pi_A(p)$ be the unique point of A that is closest to p with distance $r = \|q - p\|$. The ball $B_r(p) = \{x : \|x - p\| \leq r\}$ is a compact convex set with $B_r(p) \cap A = \{q\}$. For $c = q - p$, consider the hyperplane

$$H_0 \;=\; \{x : c^t x = c^t q\}.$$

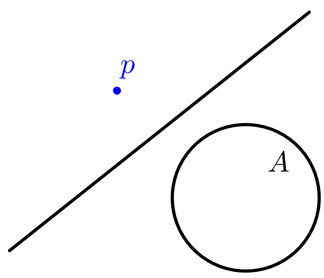
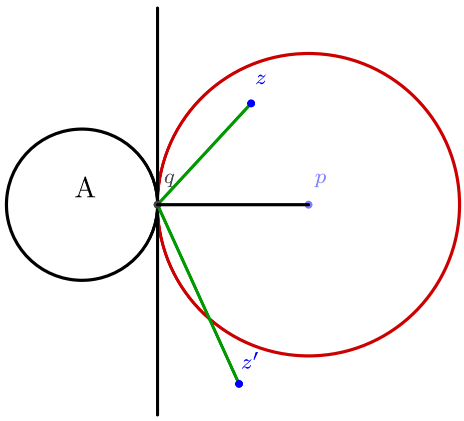


FIGURE 5. A strictly separating hyperplane

We have $p \in H_0^{<}0$ and we want to argue that $A \subseteq H_0^{\geq}$. Indeed, elementary (plane) geometry gives us that if $z \in H_0^{>}$ and $z \neq q$, then the segment $[q, z]$ meets the interior of $B_r(p)$ and hence contains a point that is closer to p than q. Since $p \neq q$, taking the hyperplane

$$H \ := \ \left\{x : c^t x = \tfrac{1}{2} c^t (p+q)\right\}$$

that is parallel to H_0 and passes through the midpoint between p and q yields the strictly separating hyperplane. □



The hyperplane H_0 constructed in the course of the proof has a special property:

DEFINITION 25 (Supporting hyperplane). A hyperplane H is called **supporting** for $A \subseteq \mathbb{R}^d$ if $A \subseteq H^-$ and $H \cap A \neq \varnothing$.

The separation theorem above implies that for a nonempty closed convex set $A \subseteq \mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, we can always find a hyperplane H which is supporting for A and which separates A and p.

r:halfspace_dec COROLLARY 26. Let $A \subseteq \mathbb{R}^d$ be a closed and convex set. Then

$$\begin{aligned} A &= \bigcap \{H^{\leq} : H \text{ hyperplane with } A \subseteq H\} \\ &= \bigcap \{H^{\leq} : H \text{ is a supporting hyperplane of } A\}. \end{aligned}$$

PROOF. To every hyperplane with $A \subseteq H^{\leq}$ there is a supporting hyperplane $\hat{H}$ parallel to H. This observation proves the second equality. By definition A is contained in the right-hand side of the first equality. To prove the reserve inclusion, let $p \in \mathbb{R}^d \setminus A$. By the Separation Theorem there is a hyperplane H strictly separating A from p. Hence p is not contained in the right-hand side which proves the claim. □

For a compact set $K \subset \mathbb{R}^d$, we define the **support function** $h_K : \mathbb{R}^d \to \mathbb{R}$ by

$$h_K(c) \;:=\; \max\{c^t x : x \in K\}.$$

A direct consequence of Corollary 26 is the following.

cor:supp_char COROLLARY 27. A convex body $K \subseteq \mathbb{R}^d$ is determined by its support-function h_K, that is,

$$K \;=\; \bigcap_{c \in \mathbb{R}^d} \{x \in \mathbb{R}^d : c^t x \le h_K(c)\}.$$

Moreover, for two convex bodies $K_1 = K_2$ if and only if $h_{K_1} = h_{K_2}$.

The support function will play an important role in later chapters. For now, we use it to give the collection of convex bodies some algebraic structure. The **Minkowski sum** of two sets $A, B \subseteq \mathbb{R}^d$ is defined as

$$A + B \;=\; \{a + b : a \in A, b \in B\}.$$

PROPOSITION 28. The Minkowski sum $A + B$ is convex whenever $A, B \subseteq \mathbb{R}^d$ are. Moreover, if $P, Q \subset \mathbb{R}^d$ are polytopes, then so is $P + Q$.

PROOF. Exercise. □

We write

$$\mathcal{K}_d \;:=\; \{K \subseteq \mathbb{R}^d : K \text{ } d\text{-convex body}\}.$$

for the **collection of convex bodies** in $\mathbb{R}^d$. Together with Minkowski addition $\mathcal{K}_d$ gets the structure of a **monoid**: The Minkowski sum $+ : \mathcal{K}_d \times \mathcal{K}_d \to \mathcal{K}_d$ is associative $(K_1 + K_2) + K_3 = K_1 + (K_2 + K_3)$ and $\{0\}$ serves as a neutral element $K_1 + \{0\} = K_1$. So, $(\mathcal{K}_d, +, 0)$ is *almost* a group – what is missing are the inverses.

prop:supp_props PROPOSITION 29. Let $K_1, K_2, L \in \mathcal{K}_d$ be convex bodies. Then

(1) $h_{K_1 + K_2} = h_{K_1} + h_{K_2}$, and
(2) If $K_1 + L = K_2 + L$, then $K_1 = K_2$.

PROOF. For $c \in \mathbb{R}^d$, let $p_i \in K_i$ such that $c^t p_i = h_{K_i}(c)$. Then $p_1 + p_2$ implies that

$$h_{K_1}(c) + h_{K_2}(c) \;=\; c^t p_1 + c^t p_2 \;=\; c^t(p_1 + p_2) \;\le\; h_{K_1 + K_2}(c).$$

Conversely, $p_1 + p_2 \in K_1 + K_2$ with $c^t(p_1 + p_2) = h_{K_1 + K_2}(c)$ shows $h_{K_1} + h_{K_2} \ge h_{K_1 + K_2}$.
For ii) we observe that it suffices to show that $h_{K_1} = h_{K_2}$. Using i), we infer

$$h_{K_1} - h_{K_2} \;=\; (h_{K_1} + h_L) - (h_{K_2} + h_L) \;=\; h_{K_1 + L} - h_{K_2 + L} \;=\; 0$$

□

1.5. Separation - More general.

There are more general separation theorems.

thm:gen_sep THEOREM 30. *Let $A, B \subseteq \mathbb{R}^d$ be convex sets.*

(i) If $\operatorname{relint}(A) \cap \operatorname{relint}(B) = \varnothing$ *then A and B can be separated.*
(ii) If A is closed and B is compact and $A \cap B = \varnothing$ then A and B can be strictly separated.

PROOF. Exercise. □

In part (ii) of the above theorem it is not sufficient for B to be closed. For example if

$$A \;=\; \{(x, y) \in \mathbb{R}^2 : x \ge 1, xy \ge 1\}$$
$$B \;=\; \{(x, y) \in \mathbb{R}^2 : y = 0\}$$

then we cannot find a hyperplane which strictly separates A from B (see picture below).

lem:bdy_supp LEMMA 31. Let $A \subseteq \mathbb{R}^d$ be a closed and convex set and $p \in \partial A$. Then there is a hyperplane H supporting for A but not containing A such that $p \in A \cap H$.

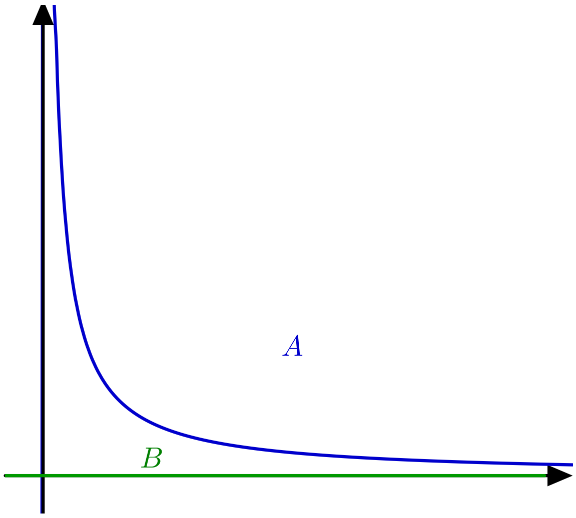


FIGURE 6. Two convex sets which cannot be strictly separated

PROOF. Assume that A is full-dimensional. Set $C = \text{int}(A)$ and $D = \{p\}$. C and D are relatively open, convex, and disjoint. Thus, by Theorem 30 there is a hyperplane $H = \{x \in \mathbb{R}^d : c^t x = \delta\}$ such that $c^t p = \delta$ and

$$c^t x \;\le\; \delta \qquad \text{for all } x \in C.$$

Hence, $C \subseteq H^{\le}$ and, since $H^{\le}$ is closed, $A = \bar{C} \subseteq H^{\le}$. □

Lecture 4, April 22

We can now prove Minkowski's Theorem (Theorem 21). Recall that $\text{ext}(K)$ is the set of extreme points of K, that is, points p that cannot be expressed as convex combinations of points in $K \setminus \{p\}$. We need one more observation.

prop:ext_supp PROPOSITION 32. Let $K \subset \mathbb{R}^d$ be a convex body and H a supporting hyperplane. Then

$$\text{ext}(K \cap H) \;=\; \text{ext}(K) \cap H.$$

PROOF. The easy direction is $\text{ext}(K \cap H) \supseteq \text{ext}(K) \cap H$. Indeed, if $p \in \text{ext}(K) \cap H$ cannot be expressed as a convex combination of elements in K, then it cannot be expressed by elements in $K \cap H$.

Let $H = \{c^t x = \delta\}$ so that $c^t x \le \delta$ for all $x \in K$. Let $p = (1-\lambda)p_1 + \lambda p_2$ with $0 \le l_1 \le 1$ and $p_1, p_2 \in K$. If $p \in H$, then $p_1, p_2 \in H$. Indeed,

$$\delta \;=\; c^t p \;=\; (1-l)\underbrace{c^t p_1}_{\le\delta} + \lambda \underbrace{c^t p_2}_{\le\delta} \;\le\; \delta$$

with equality if and only if $c^t p_1 = c^t p_2 = \delta$. Hence $p \in \text{ext}(K \cap H)$ implies $p \in \text{ext}(K) \cap H$. □

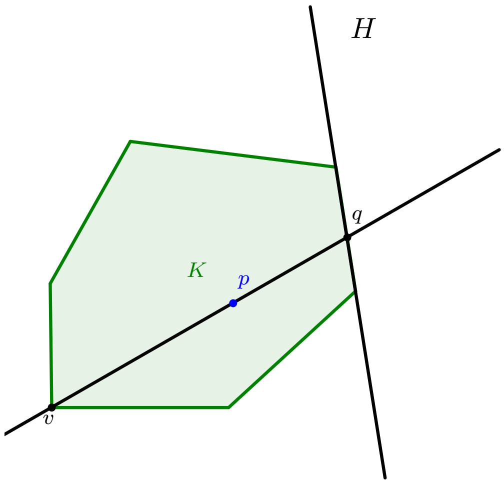
PROOF OF MINKOWSKI'S THEOREM. Since $\text{ext}(K) \subseteq K$, we have $\text{conv}(\text{ext}(K)) \subseteq K$ and thus we only have to prove the reverse inclusion. The proof is by induction on the dimension $d = \dim(K)$.

If $d = 0$, then K is a point and the theorem is trivially true. For $d = 1$, K is a segment and thus the convex hull of its two endpoints. Hence for $d = 1$ Minkowski's theorem also is true.

Thus, we assume that the assertion is true for all convex bodies of dimension $< d$. Let $K \subset \mathbb{R}^d$ be a full-dimensional convex body and $p \in K$.

If $p \in \partial K$, then there is a supporting hyperplane H with $p \in K \cap H$. In particular $F = K \cap H$ is a convex body of dimension $\le \dim K - 1$ and by induction $p \in \text{conv}(\text{ext}(K \cap H))$. By Proposition 32, we have $\text{ext}(K \cap H) \subseteq \text{ext}(K)$ and we are done.

Assume that $p \in \operatorname{int}(K)$. Pick $v \in \operatorname{ext}(K)$ and consider the affine line $\overline{vp}$. It meets ∂K in two points v and q. The point q is in the boundary of K and hence $q \in \operatorname{conv}(V)$ for some $V \subseteq \operatorname{ext}(K)$ and we conclude that $p \in \operatorname{conv}(V \cup \{v\}) \subseteq \operatorname{conv}(\operatorname{ext}(K))$ which completes the proof.



□

Definition 33 (Face). Let K be a convex set. A convex subset $F \subseteq K$ is called a **face** if for all $x, y \in K$

$$(x, y) \cap F \neq \emptyset \;\Rightarrow\; x, y \in F.$$

An **exposed** face is a set of the form $G = K \cap H$ for a hyperplane H such that .

Note that $\emptyset$ and K are faces of K and we *define* that $\emptyset$ and K are also exposed faces. This can be justified by the wish to make the notion of 'face' to be independent of the embedding. Now embedding K into a hyperplane $H \subset \mathbb{R}^{d+1}$ trivially allows us to get $\emptyset$ and K as faces.

Every face $F \neq K$ is called a **proper** face. Faces of dimension 0 are exactly the extreme points of K. If $K \cap H = \{p\}$ for some supporting hyperplane, then p is called an **exposed point** of K.

Proposition 34. Let $K \subseteq \mathbb{R}^d$ be a closed convex set.

(i) Every exposed face is a face.
(ii) If $F, G \subseteq K$ are faces of K, then $F \cap G$ is a face of K.
(iii) If $F \subseteq K$ face is a K and $G \subseteq F$ is a face of F, then G is a face of K.

Proof.

(i) Let $H = \{x \in \mathbb{R}^d : c^t x = \delta\}$ be a supporting hyperplane for K and $F = H \cap K$. Assume that $p = (1-\lambda)x + y\lambda \in F$ with $x, y \in K$ and $0 < \lambda < 1$. Since $p \in H$, we calculate

$$\delta \;=\; c^t p \;=\; (1-\lambda)\underbrace{c^t x}_{\leq \delta} + \lambda \underbrace{c^t y}_{\leq \delta} \;\leq\; \delta$$

which implies $x, y \in H$ and hence $x, y \in F$.
(ii) You do the argument!
(iii) Assume that $p = (1-\lambda)x + \lambda y \in G$ for $1 < \lambda < 1$ and some $x, y \in K$. Since $p \in F$ and F is a face of K, it follows that $x, y \in F$. Likewise, since G is a face of F, it follows that $x, y \in G$. □

The converse to (i) is generally not true: The points p and q of the convex hull of the "stadium curve" are (two of the four) non-exposed points of K.

The following lemma shows that the relative interiors of faces of a convex body K give a partition of K.

lem:rel_faces

Lemma 35. Let K be a closed convex set.

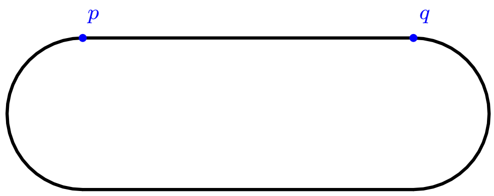
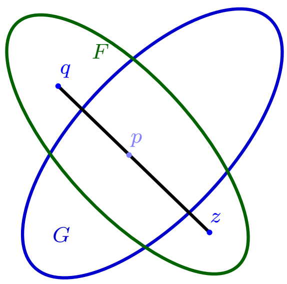


FIGURE 7. A convex body with non-exposed faces p and q

(i) Let $F, G \subseteq K$ be distinct faces. Then $\operatorname{relint}(F) \cap \operatorname{relint}(G) = \varnothing$.
(ii) For every non-empty, convex, and relatively open set $A \subseteq K$ there is a unique face $F_A \subseteq K$ such that $A \subseteq \operatorname{relint}(F_A)$. In particular every point $p \in K$ lies in the relative interior of a unique face F_p.

PROOF.

(i) Assume $p \in \operatorname{relint}(F) \cap \operatorname{relint}(G)$. Without loss of generality $G \nsubseteq F$ and let $q \in G \setminus F$. Since $p \in \operatorname{relint}(G)$, there exists a point $z \in G$ such that $p = (1-\lambda)q + \lambda z$ with $0 < \lambda < 1$. Now, since F is a face, we have $q \in F$ which is a contradiction.



(ii) The set
$$F_A \;:=\; \bigcap \{G \subseteq K : G \text{ face of } K, A \subseteq G\}.$$
is the inclusion-minimal face of K that contains A. We need to check $A \subseteq \operatorname{relint}(F_A)$.

Suppose there is $p \in A \setminus \operatorname{relint}(F_A)$. Then $p \in \partial F_A$ and by Lemma 31 there is a hyperplane H supporting but not containing F_A such that $p \in G := F_A \cap H$. This, in particular, means that G is a *proper* (exposed) face of F_A. We want to argue that $A \subseteq G$. Indeed, if $q \in A \setminus G$, then, since A is relatively open, there is a point $z \in A$ such that $p \in (q, z)$. Since $q, z \in A \subseteq F_A$ and G is a face of F_A, it follows that $q, z \in G$. This contradiction implies $A \subseteq G$. But G is a face of F_A and hence a face of K. Consequently, $F_A \subseteq G$ by definition of F_A, which contradicts the fact that G is a proper face of F_A. Hence, there is no point $p \in A \setminus \operatorname{relint}(F_A)$ which proves the claim. □

Faces form a partially ordered set under the inclusion known as the **face lattice**
$$\mathcal{F}(K) \;:=\; \{F \subseteq K : F \text{ is a face of } K\}.$$
We know that $\mathcal{F}(K)$ has a minimum ($\varnothing$) and a maximum (K) and that for every two elements $F_1, F_2 \in \mathcal{F}$ the face $F_1 \cap F_2$ is the unique inclusion-maximal face contained in both F_1 and F_2. Conversely, Lemma 35 (ii) implies that there is a unique, inclusion-minimal face F containing $\operatorname{relint}(\operatorname{conv}(F_1 \cup F_2))$. In the language of posets, this says that $\mathcal{F}(K)$ is a **lattice** with **meet** $F_1 \wedge F_2 := F_1 \cap F_2$ and **join** $F_1 \vee F_2 := F$. For polytopes this partially ordered set carried a lot of structure. For general convex bodies this, unfortunately, is not the case: Look for example at the unit ball $\mathcal{F}(B_d)$.

In contrast to polytopes:

- There is not necessarily a face $F \subseteq K$ of dimension i for every $0 \leq i \leq \dim K$. For example the disc B_2.
- In particular $\mathcal{F}(K)$ is not necessarily graded. For example the half-disc $B_2 \cap \{x : x_1 \geq 0\}$.
- Let $F \subseteq K$ be an exposed face of K and $G \subseteq F$ an exposed face of F. Then $G \subseteq K$ is not necessarily an exposed face of K. See the picture below for an example.

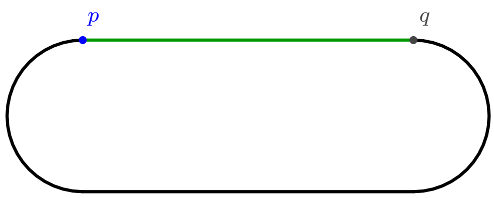


FIGURE 8. The green segment is an exposed face, but p and q are not.

- Let be an $F \subseteq K$ exposed face of K and $F \subseteq G \subseteq K$ face, then G is not necessarily exposed.

Life is good with polytopes.

PROPOSITION 36. Every face of a polytope is a polytope. In particular, every face of a polytope is exposed.

PROOF. For every face $F \subseteq K$, we have $\mathrm{ext}(F) \subseteq \mathrm{ext}(K)$. Hence, $\mathrm{ext}(K)$ is finite, then so is $\mathrm{ext}(F)$.
You(!) prove the other claim. □

1.6. Polarity/Duality.

DEFINITION 37. For $A \subseteq \mathbb{R}^d$ we define the **polar** of A as

$$A^{\triangle} := \{y \in \mathbb{R}^d : y^t x \leq 1 \text{ for all } x \in A\}$$

We note that $A^{\triangle}$ is a closed, convex set with $0 \in A^{\triangle}$. Indeed, for $x \in A$, let us write $H_x = \{y \in \mathbb{R}^d : x^t y = 1\}$. Then

$$A^{\triangle} = \bigcap_{x \in A} H_x^{\leq}$$

is an intersection of closed convex halfspaces all containing the origin.
Let us record some basic properties.

PROPOSITION 38.

(i) If $L \subseteq \mathbb{R}^d$ is a linear subspace, then $L^{\triangle} = L^{\perp}$.
(ii) If $A \subseteq B$, then $A^{\triangle} \supseteq B^{\triangle}$.
(iii) $(\bigcup_{i \in I} A_i)^{\triangle} = \bigcap_{i \in I} A_i^{\triangle}$ for a family $(A_i)_{i \in I}$ of convex sets.
(iv) If $P = \mathrm{conv}(v_1, \ldots, v_n)$, then

$$P^{\triangle} = \{y \in \mathbb{R}^d : v_i^t y \leq 1, i = 1, \ldots, n\}.$$

$P^{\triangle}$ is a **polyhedron**, that is, the intersection of finitely many halfspaces.
(v) $A \subseteq (A^{\triangle})^{\triangle}$

PROOF. You(!) do the proof. □

For a finite intersection of halfspaces we also write $P = \{x : Ax \leq b\}$ for some $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ and $x \leq y$ means $x_i \leq y_i$ for all i. The most important property is given by the following result.

THEOREM 39 (Bipolar theorem). *Let $A \subseteq \mathbb{R}^d$ be closed and convex with $0 \in A$. Then*

$$(A^{\triangle})^{\triangle} = A.$$

Lecture 5, April 28

Proof. Since $A \subseteq (A^{\triangle})^{\triangle}$ by the previous proposition, we only need to show $A \supseteq (A^{\triangle})^{\triangle}$. So, let $z \in (A^{\triangle})^{\triangle} \setminus A$. Since A is closed and $\{z\}$ is compact there is a strictly separating hyperplane, i.e., there is some $y \in \mathbb{R}^d \setminus \{0\}$ and $\delta \in \mathbb{R}$ such that $y^t x < \delta$ for all $x \in A$ and $y^t z > \delta$. Since $0 \in A$ we infer that $\delta > 0$ and $\bar{y} := \frac{1}{\delta} y$ satisfies

$$\bar{y}^t z > 1 \qquad \text{and} \qquad \bar{y}^t x < 1 \text{ for all } x \in A$$

Now the later implies $\bar{y} \in A^{\triangle}$. The inequality $z^t \bar{y} > 1$ with $\bar{y} \in A^{\triangle}$ implies $z \notin (A^{\triangle})^{\triangle}$ which contradicts our assumption $z \in (A^{\triangle})^{\triangle}$. □

We conclude that for general $A \subseteq \mathbb{R}^d$ that $(A^{\triangle})^{\triangle} = \overline{\mathrm{conv}(A \cup \{0\})}$.

Proposition 40. Let $A \subseteq \mathbb{R}^d$ be any set.

(i) $(\alpha A)^{\triangle} = \frac{1}{\alpha} A^{\triangle}$ for $\alpha \neq 0$.
(ii) $A = A^{\triangle}$ if and only if $A = B_d$ is the unit ball.

Proof. Do it yourself. □

Corollary 41. Let A be closed and convex. Then A is bounded if and only if $0 \in \mathrm{int}(A^{\triangle})$.

Proof. A is bounded if only only if $A \subseteq rB_d$ for some $r > 0$. From the previous proposition we infer that

$$A^{\triangle} \supseteq \tfrac{1}{r} B_d$$

which implies that $0 \in \mathrm{int}(A^{\triangle})$. Reversing this argument completes the proof. □

Theorem 42 (Minkowski-Weyl Theorem – Polytope version). *Let $K \subseteq \mathbb{R}^d$ be a convex body with $0 \in \mathrm{int}(K)$ (in particular, K is non-empty and full-dimensional). Then K is a polytope if and only if $K^{\triangle}$ is a polytope.*

Proof. We only need to show that if K is a polytope then $K^{\triangle}$ is a polytope. For the converse, we use polarity to conclude that if $K^{\triangle}$ is a polytope, then $K = (K^{\triangle})^{\triangle}$ is a polytope. We will show that K is the intersection of finitely many halfspaces, i.e.,

$$K \;=\; \{x \in \mathbb{R}^d : a_i^t x \leq 1, i = 1, \ldots, m\} \;=\; \bigcap_{i=1}^{m} \{x \in \mathbb{R}^d : a_i^t x \leq 1\}$$

for some $a_1, \ldots, a_m \in \mathbb{R}^d \setminus 0$. If we have that, we can conclude from polarity that

$$K^{\triangle} \;=\; \mathrm{conv}(a_1, \ldots, a_m).$$

and $K^{\triangle}$ is a polytope.

Since K is a polytope, there is some $V = \{v_1, \ldots, v_n\}$ such that $K = \mathrm{conv}(V)$. Let us call an (oriented) hyperplane $H \subset \mathbb{R}^d$ a *V-hyperplane* if

(i) $\mathrm{aff}(V \cap H) = H$ and
(ii) $K \subseteq H^{\leq}$.

The first property means that H is spanned by a proper subset $V' \subset V$ and, in particular, can be identified with $V' = H \cap V$. Thus, the set of V-hyperplanes is finite and we let $H_1, \ldots, H_m$ be this collection of hyperplanes. The second property states that every H_i is supporting for K and we define

$$K' \;:=\; \bigcap_{i=1}^{m} H_i^{\leq}.$$

We claim that $K = K'$. By construction $K \subseteq K'$. Hence, we only need to show that $p \notin K$ implies $p \notin K'$. We want to find a point $q \in \mathrm{int}\, K$ such that p and q are strictly separated by some V-hyperplane H_i. By choosing q wisely, we can find H_i as the unique supporting hyperplane that contains $\partial K \cap [p, q]$.

Let us consider

$$R := \bigcup\{\text{aff}(F) : F \subset K \text{ face}, \dim F \le d-2\},$$

By Proposition 32, this is a finite union of affine subspaces of dimensions $\le d-2$. Make sure to verify that

$$\{q \in \text{int}\, K : (p,q) \cap R = \varnothing\} \ne \varnothing$$

and let pick q be a point from. Now, let $r \in \partial K \cap [p,q]$. There is a supporting hyperplane H with $r \in K \cap H$ and $F = K \cap H$ is a face. Now, by construction F is a proper face of dimension $> d-2$, hence of dimension $d-1$. This implies that $H = \text{aff}(F) = \text{aff}(V \cap H)$ and $H = H_i$ for some i. In particular, H_i separates p from K' and hence $p \notin K'$. □

We get some nice consequences.

Corollary 43. A convex set $P \subseteq \mathbb{R}^d$ is polytope if and only if P is a bounded polyhedron, i.e. a finite intersection of halfspaces which is bounded.

Corollary 44. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope. Then, up to positive scaling, there are unique $a_1, \dots, a_m \in \mathbb{R}^d \setminus \{0\}$ and $b_1, \dots, b_m \in \mathbb{R}$ such that

$$P = \{x \in \mathbb{R}^d : a_i^t x \le b_i, i = 1, \dots, m\}$$

Proof. By translating if necessary, we can assume that $0 \in \text{int}(P)$. Hence $0 = a_i^t 0 < b_i$ for all i and we may assume that $b_i = 1$. Thus

$$P^\triangle = \text{conv}(a_1, \dots, a_m).$$

and the result follows from the fact that $\text{ext}(P^\triangle) \subseteq \{a_1, \dots, a_m\}$ is the inclusion minimal set with $P^\triangle = \text{conv}(\text{ext}(P^\triangle))$. □

Let $P = \{x \in \mathbb{R}^d : a_i^t x \le b_i, i = 1, \dots, m\}$ be a polyhedron. An inequality $a_i^t x \le b_i$ for some i is **irredundant** if

$$P \ne \bigcap_{j \ne i} \{x : a_j^t x \le b_j\}.$$

The corollary states that every polytope has an irredundant description as an intersection of halfspaces.

A proper face $F \subset K$ of dimension $\dim K - 1$ is called a **facet**.

Proposition 45. Let $K \subset \mathbb{R}^d$ be a closed convex set.

(i) Every proper face is contained in an exposed face.
(ii) Every facet is exposed.
(iii) Let $P = \{x \in \mathbb{R}^d : a_i^t x \le b_i, i = 1 \dots m\}$ be a polytope such that all inequalities are irredundant. Then $F \subseteq P$ is a facet if and only if $F = P \cap \{x \in \mathbb{R}^d : a_i^t x = b_i\}$ for some i.

Proof. For (i) let $G \subset K$ be a proper face and let $p \in \text{relint}\, G$. Since $p \in \partial K$, by Lemma 31 there is a supporting hyperplane H such that $p \in F := K \cap H$. To show that $G \subseteq F$, assume that $q \in G \setminus F$. Then, since $p \in \text{relint}\, G$, there is some $z \in G$ with $p \in (q,z)$. Since $p \in F$, this implies $q, z \in F$.

This argument also shows that $\dim G \le \dim F < \dim K$. Hence if G is a facet, then $F = G$ which proves (ii).

(iii): Do it yourself! □

Lecture 6, April 29

Definition 46. Let $K \subseteq \mathbb{R}^d$ be a convex body with $0 \in \text{int}(K)$ and $F \subseteq K$ a face of K. The **conjugate** of F is defined as

$$F^\diamond := \{y \in K^\triangle : y^t x = 1 \text{ for all } x \in F\}.$$

Proposition 47. Let $K \subseteq \mathbb{R}^d$ be a convex body.

(i) $\varnothing^\diamond = K^\triangle$, $K^\diamond = \varnothing$
(ii) If $G \subseteq G \subseteq K$, then $F^\diamond \supseteq G^\diamond$.
(iii) If $F \subseteq K$ is a proper face of K, then $F^\diamond$ is an exposed face of $K^\triangle$.
(iv) For a set $A \subseteq K$, $(A^\diamond)^\diamond$ is the inclusion-minimal exposed face of K containing F.

PROPOSITION 48. Let $P \subseteq \mathbb{R}^d$ be a polytope with $0 \in \operatorname{int}(P)$ and $F \subseteq P$ a (proper) face of P. Then

$$\dim(F) + \dim(F^o) \;=\; \dim(P) - 1.$$

PROOF. We know that

$$\dim(F) = \dim(\operatorname{aff}(F)) \;=\; \dim(U)$$

for some linear subspace U of $\mathbb{R}^d$. Also

$$\dim(F^\diamond) \;=\; \dim(\operatorname{aff}(F^\diamond)) \;=\; \dim(U^\perp).$$

Thus

$$\dim(F) + \dim(F^\diamond) \;=\; \dim(P) - 1.$$

□

The above proposition does not hold for general convex bodies. A simple counterexample is the unit disk.

Recall that a set $C \subseteq \mathbb{R}^d$ is a **convex cone**, if $\lambda x + \mu y \in C$ for all $x, y \in C$ and $\lambda, \mu \geq 0$. Like the convex hull, we have for the **conical hull** of $S \subset \mathbb{R}^d$

$$\operatorname{cone}(S) \;:=\; \bigcap \{C \text{ convex cone} : S \subseteq C\} \;=\; \{\mu_1 s_1 + \cdots + \mu_k s_k : s_1, \ldots, s_k \in S, \mu_1, \ldots, \mu_k \geq 0\}$$

We call a cone C **finitely generated** if $C = \operatorname{cone}(S)$ for some fintie S. A simple observation is that the intersection of linear halfspaces is always a convex cone. A set C is a **polyhedral cone** if

$$C \;=\; \{x \in \mathbb{R}^d : a_i^t x \leq 0 \text{ for } i = 1, \ldots, m\}$$

for some $a_1, \ldots, a_m \in \mathbb{R}^d$.

PROPOSITION 49. Let $C \subset \mathbb{R}^d$ be a convex cone. Then

$$C^\triangle \;=\; \{y \in \mathbb{R}^d : y^t x \leq 0 \text{ for all } x \in C\}$$

Hence, the polar of a convex cone is a convex cone.

PROOF. The inclusion $\supseteq$ is clear. Suppose that for some $y \in C^\triangle$ there is a point $x \in C$ such that $y^t x > 0$. Then $y^t(\mu x) > 1$ for some $\mu > 0$ and, since $\mu x \in C$, we get the contradiction $y \notin C^\triangle$. □

The Bipolar theorem also helps in obtaining the following characterization.

THEOREM 50 (Minkowski-Weyl – cone version). *Let $C \subseteq \mathbb{R}^d$. Then C is a polyhedral cone if and only if C is a finitely generated cone.*

We close the section on polarity with a result that is very useful in practise.

lem:farkas_cone LEMMA 51 (Farkas lemma – cone version). Let $A \in \mathbb{R}^{d \times n}$ and $b \in \mathbb{R}^d$. Then exactly one of the two conditions holds.

(1) There is a point $x \in \mathbb{R}^m$ such that

$$Ax \;=\; b \qquad \text{and} \qquad x \geq 0.$$

(2) There is some $y \in \mathbb{R}^d$ such that

$$y^t A \leq 0 \qquad \text{and} \qquad y^t b > 0$$

PROOF. The columns of A are denoted by $a_1, \ldots, a_m$. Then condition (1) simply says $b \in \operatorname{cone}(a_1, \ldots, a_m) =: C$. This is the case if and only if $y^t b \leq 0$ for all $y \in C^\triangle = \{y : y^t a_i \leq 0, i = 1, \ldots, m\}$. □

m:farkas_affine

LEMMA 52 (Farkas lemma – affine version). Let $A \in \mathbb{R}^{n\times d}$ and $b \in \mathbb{R}^n$. Then exactly one of the two conditions holds.

(1) There is a point $x \in \mathbb{R}^d$ such that

$$Ax \le b$$

(2) There is some $y \in \mathbb{R}^m$ with $y \ge 0$ such that

$$y^t A = 0 \quad \text{and} \quad y^t b < 0$$

PROOF. Scaling by a positive number if necessary, we can assume that $y^t b = -1$. Hence (2) is equivalent to the existence of a $y \in \mathbb{R}^m$ such that $y \ge 0$ and

$$\begin{pmatrix} A^t \\ -b^t \end{pmatrix} y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By the cone version of Farkas lemma, if there is no such solution, then there is $\binom{x}{\alpha} \in \mathbb{R}^{d+1}$ such that

$$(x^t, \alpha)\begin{pmatrix} A^t \\ -b^t \end{pmatrix} \le 0 \quad \text{and} \quad (x^t,\alpha)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha > 0$$

By rescaling if necessary, we can assume that $\alpha = 1$ and hence, the condition is $Ax - b \le 0$. □

We digress for a moment to give an important application of the Farkas lemma. A **linear program** is the task to find a solution to

$$\text{(P)} \qquad \begin{array}{ll} \max & c^t x \\ \text{subject to} & Ax \le b \end{array}$$

for some $A \in \mathbb{R}^{n\times d}, b \in \mathbb{R}^n, c \in \mathbb{R}^d$. This is called the **primal** linear program. The associated **dual** linear program is

$$\text{(D)} \qquad \begin{array}{ll} \min & y^t b \\ \text{subject to} & y^t A = c^t \\ & y \ge 0 \end{array}$$

Assume that both programs are feasible, i.e., there is x and y satisfying the linear constraints. Then we calculate

$$c^t x = (y^t A)x = y(Ax) \le y^t b.$$

In particular, if c_* and b_* are the optimal values for the primal and dual linear program then $c_* \le b_*$. This is called **weak duality**. For linear programming, we even have a strong form of duality.

THEOREM 53 (Strong LP-duality). *Assume that both (P) and (D) are feasible with optimal values c_* and b_*, then $c_* = b_*$.*

PROOF. By contradiction, we assume that the system

$$\begin{pmatrix} -c^t \\ A \end{pmatrix} x \le \begin{pmatrix} -b_* \\ b \end{pmatrix}$$

does not have a solution. By Lemma 52, there is then a vector $(\alpha, y) \ge 0$ such that

$$y^t A = \alpha c^t \quad \text{and} \quad y^t b < \alpha b_*$$

If $\alpha = 0$, then this states that (P) is infeasible. Hence, we can assume that $\alpha = 1$. But then, this states that y is a solution to (D) with value $y^t b < b_*$. □

There are numerous applications of strong duality. The easiest one is that strong duality enables us to obtain a *certificate* for the optimality of a solution: If x and y are solutions to (P) and (D), respectively, such that $c^t x = y^t b$, then x and y are optimal.

1.7. The support function. For a non-empty convex body $K \subseteq \mathbb{R}^d$ the **support function** was defined as

$$h_K(c) \;:=\; \max\{c^t x : x \in K\}.$$

The support function has the following two nice properties.

PROPOSITION 54. Let $K \neq \varnothing$ be a convex body with support function h_K.

(i) h_k is **positive linear** or **positive homogeneous**: For $c \in \mathbb{R}^d$ and $\lambda \geq 0$

$$h_K(\lambda c) \;=\; \lambda h_K(c).$$

(ii) h_K is **subadditive**: For $c_1, c_2 \in \mathbb{R}^d$

$$h_K(c_1 + c_2) \;\leq\; h_K(c_1) + h_K(c_2).$$

PROOF. (i) is clear from the definition. For (ii) let $x \in K$ such that $(c_1+c_2)^t x = h_K(c_1+c_2)$. Then

$$h_K(c_1+c_2) \;=\; (c_1+c_2)^t x \;=\; c_1^t x + c_2^t x \;\leq\; h_K(c_1) + h_K(c_2).$$

$\square$

A function $f : \mathbb{R}^d \to \mathbb{R}$ that satisfies both properties is called **sublinear**. Both properties in particular imply that such an f is *convex* and continuous!

These two properties give a characterization of support functions.

thm:supp_fct THEOREM 55. *If $f : \mathbb{R}^d \to \mathbb{R}$ is a sublinear function, then there is a unique convex body $K \subset \mathbb{R}^d$ such that $f = h_K$.*

PROOF. Uniqueness follows from Corollary 27. We define

$$K \;:=\; \{x \in \mathbb{R}^d : c^t x \leq f(c) \text{ for all } c \in \mathbb{R}^d\}.$$

By definition K is a closed convex set. Let $M := \max\{f(c) : \|c\| = 1\}$. If K is unbounded, then there is a point $u \in K$ with $\|u\| > M$. But then for $c = \frac{1}{\|u\|}c$, we get $\|u\| = c^t u \leq f(c) \leq M$.

Now, if $K \neq \varnothing$, then $h_K(c) \leq f(c)$ for all c. So it suffices to show that $K \neq \varnothing$ and $h_K(c) \geq f(c)$ for all c.

Recall that the epigraph of a function f is the set

$$C \;:=\; \operatorname{epi}(f) \;=\; \{(x,t) \in \mathbb{R}^{d+1} : t \geq f(x)\}.$$

Since f is convex and positively homogeneous, $\operatorname{epi}(f)$ is a closed convex cone. For $c \in \mathbb{R}^d$, $(c, f(c)) \in \partial C$. Indeed, for every $\varepsilon > 0$, $(c, f(c)+\varepsilon) \in C$ whereas $(c, f(c)-\varepsilon) \notin C$. By Lemma 31, there is a supporting hyperplane H for C such that $(c, f(c)) \in H \cap C$. The supporting hyperplane is of the form $H = \{(x,t) : y^t x + \alpha t = 0\}$.

Since $(u,s) \in C$ implies that $(u, s+a) \in C$ for all $a \geq 0$, we get that $\alpha \leq 0$. If $\alpha = 0$, then $(c, f(c)) \in C \subset H^{\leq}$ implies $y^t c \leq 0$ for all $c \in \mathbb{R}^d$ and hence $y = 0$. Thus, $\alpha < 0$ and we can assume $\alpha = -1$.

This implies $u^t y \leq f(u)$ for all $u \in \mathbb{R}^d$ and hence $y \in K$. So K is not empty. Moreover, $c^t y = f(c)$ which implies $f(c) \leq h_K(c)$. $\square$

The gist of the proof is the following: If f is convex, then $C := \operatorname{epi}(f)$ is a convex set. Since f is positively linear, C is even a convex cone. The polar $C^{\triangle}$ is a closed and convex cone and the set K is given by

$$K \;:=\; \{x \in \mathbb{R}^d : (x,-1) \in C^{\triangle}\} \;=\; C^{\triangle} \cap \{(x,t) : t = -1\}.$$

The vector space $C^0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$, the convex functions form a convex cone. In particular,

$$\mathcal{L}_d \;:=\; \{f : \mathbb{R}^d \to \mathbb{R} : f \text{ sublinear}\}$$

is a convex cone. Theorem 55 states that

$$\mathcal{K}_d \;\cong\; \mathcal{L}_d$$

under the correspondence $K \mapsto h_K$. However, for $K_1, K_2 \in \mathcal{K}_d$ and $\mu_1, \mu_2 \geq 0$ we know that

$$h_{\mu_1 K_1 + \mu_2 K_2} = \mu_1 h_{K_1} + \mu_2 h_{K_2}.$$

Hence, we can view $\mathcal{K}_d$ is isomorphic to a closed convex cone.

Lecture 7, May 5

1.8. The distance function. Let $K \subseteq \mathbb{R}^d$ be a full-dimensional convex body with $0 \in \operatorname{int} K$. In Example 2 we argued that if K is centrally-symmetric ($-K = K$), then K induces a norm on $\mathbb{R}^d$. If K is not centrally-symmetric, this yields only a semi-norm together with a **distance function**: For $x \in \mathbb{R}^d$, we define

$$d_K(x) := \min\{\lambda \geq 0 : x \in \lambda K\}.$$

The following result gives a nice description of distance functions.

thm:dist_fct THEOREM 56. *Let $K \subseteq \mathbb{R}^d$ be a full-dimensional convex body with $0 \in \operatorname{int}(K)$, then*

$$d_k = h_{K^\triangle}.$$

PROOF. Obviously, we have $d_K(0) = h_{K^\triangle}(0) = 0$. For every $y \in \mathbb{R}^d \setminus \{0\}$, we have that $d_K(y)$ is the smallest number $\delta > 0$ such that $\frac{1}{\delta} y \in K$. Using the Bipolar theorem, this is the smallest δ such that for all $x \in K^\triangle$

$$\tfrac{1}{\delta} y^t x \leq 1 \quad \Leftrightarrow \quad y^t x \leq \delta$$

and there is a $x_0 \in K^\triangle$ that attains equality. Viewing y as a linear function on $K^\triangle$ this says that

$$\delta = y^t x_0 = \max\{y^t x : x \in K^\triangle\} = h_{K^\triangle}(y). \qquad \square$$

COROLLARY 57. $g : \mathbb{R}^d \to \mathbb{R}$ is a distance function of some convex body K if and only if g is nonnegative and sublinear.

How to email a convex body? If we know make a census of how to represent a convex body $K \subset \mathbb{R}^d$ (with $0 \in \operatorname{int} K$), we know that we have to *remember* one of the following objects:

- the nearest point map $\pi_K : \mathbb{R}^d \to K$;
- the extreme points $\operatorname{ext}(K)$;
- the supporting hyperplanes $\operatorname{ext}(K^\triangle)$;
- the support function $h_K : \mathbb{R}^d \to \mathbb{R}$
- the distance function $d_K : \mathbb{R}^d \to \mathbb{R}$

Although all representations let us recover K, some are more suitable for certain tasks then others:

- Optimize a linear function $\ell(x) = c^t x$ over K
  Here h_K and, if K is a polytope, $\operatorname{ext}(K)$ trivially solve the problem
- Determine if $p \in K$
  Now $d_K(p) \leq 1$, $\pi_K(p) = p$, or, if K is a polytope, $\operatorname{ext}(K^\triangle)$ are of help.

In *practice* the convex bodies that we work with are polytopes that are given either in terms of vertices or inequalities. General convex body can be given by a **membership/separation oracle**. This is a *black box* (e.g. computer program) that for every $p \in \mathbb{R}^d$ either confirms that $p \in K$ or returns a hyperplane (strictly) separating p from K. Sometimes one gets only something weaker, a **weak membership/separation oracle**. For every $p \in \mathbb{R}^d$ and $\varepsilon > 0$, the oracle confirms that p is ϵ-close to K (i.e. $p \in K_\varepsilon$; see next section) or gives a separating hyperplane.

1.9. A metric on convex bodies. For a convex body $K \subset \mathbb{R}^d$ and $\varepsilon \geq 0$, we define the **outer parallel body** K_ε as

$$K_\varepsilon \ := \ \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \leq \varepsilon\}.$$

We can express this in terms of Minkowski sums.

PROPOSITION 58. $K_\varepsilon = K + \varepsilon B_d$.

PROOF. For $x \in K_\varepsilon$, if $y \in K$ is such that $\operatorname{dist}(x,y) \leq \varepsilon$, then $u := x - y \in \varepsilon B_d$ and $x = y + u \in K + \varepsilon B_d$. Conversely, for $y = x + u \in K + \varepsilon B_d$, we have $\operatorname{dist}(y, K) \leq \operatorname{dist}(y, x) \leq \varepsilon$. □

The **Hausdorff distance** of two convex bodies $K, L \subseteq \mathbb{R}^d$ is

$$\mathrm{d}(K, L) \ := \ \min\{\varepsilon \geq 0 : K \subseteq L_\varepsilon, L \subseteq K_\varepsilon\}.$$

THEOREM 59. $\mathrm{d}(\cdot,\cdot)$ *defines a metric on* $\mathcal{K}_d$.

PROOF. By definition $\mathrm{d}(\cdot,\cdot)$ is symmetric, non-negative, and $\mathrm{d}(K,L) = 0$ implies $K = L$. It remains to show that the triangle inequality holds: Let $K, L, M \subseteq \mathbb{R}^d$ be convex bodies and put $\mathrm{d}(K,L) = \alpha$ and $\mathrm{d}(L,M) = \beta$. By definition of the Hausdorff distance

$$K \ \subseteq \ L + \alpha \cdot B_d \text{ and } L \ \subseteq \ M + \beta \cdot B_d\,.$$

Thus we get

$$K \ \subseteq \ M + \alpha \cdot B_d + \beta \cdot B_d \ = \ M + (\alpha + \beta) B_d$$

and similarly

$$M \ \subseteq \ K + (\alpha + \beta) B_d$$

which implies

$$\mathrm{d}(K, M) \leq \alpha + \beta.$$

□

In the exercises you proved that $h_{B_d}(c) = \|c\|$. Together with Proposition 29, we get

$$h_{K_\varepsilon}(c) \ = \ h_{K + \varepsilon \cdot B_d}(c) \ = \ h_K(c) + \varepsilon \|c\|.$$

PROPOSITION 60. Let $K, L \subseteq \mathbb{R}^d$ be two convex bodies. Then

$$\mathrm{d}(K, L) \ = \ \max_{u \in S^{d-1}} |h_K(u) - h_L(u)|.$$

PROOF. For convex bodies $K, L \subset \mathbb{R}^d$, we have

$$\begin{aligned} L \ \subseteq K_\varepsilon \ &\Leftrightarrow \ h_L(c) \leq h_K(c) + \varepsilon \|c\| \qquad \text{for all } c \in \mathbb{R}^d \\ &\Leftrightarrow \ h_L(c) - h_K(c) \leq \varepsilon \qquad \text{for all } c \in S^{d-1} \end{aligned}$$

where the last equivalence follows from the positive homogeneity of the support function. Now $K \subseteq L_\varepsilon$ and $L \subset K_\varepsilon$ then imply $|h_K(c) - h_L(c)| \leq \varepsilon$ for all $c \in S^{d-1}$. □

Every sublinear function $f : \mathbb{R}^d \to \mathbb{R}$ restricts to a function on the unit sphere S^{d-1}. Conversely, every function $f : S^{d-1} \to \mathbb{R}$ can be uniquely extended to a positively homogeneous function $\hat{f} : \mathbb{R}^d \to \mathbb{R}$ by $\hat{f}(x) := \|x\| f(\frac{x}{\|x\|})$ for $x \neq 0$. Thus, we can identify

$$\mathcal{L}_d \ \cong \ \{f : S^{d-1} \to \mathbb{R} : \hat{f} \text{subadditive}\}.$$

We can equip $C^0(S^{d-1})$ with a norm given by

$$\|f\|_\infty \ := \ \max\{|f(p)| : p \in S^{d-1}\}.$$

This induces a metric on $\mathcal{L}_d \subset C^0(S^{d-1})$.

COROLLARY 61. The convex bodies $\mathcal{K}_d$ in $\mathbb{R}^d$ together with the Hausdorff distance are isometric to $\mathcal{L}_d$ with the ∞-metric.

1.10. Approximation by polytopes.

thm:cv_approx

THEOREM 62. *Let $K \subseteq \mathbb{R}^d$ be a non-empty convex body and $\varepsilon > 0$. Then there is a polytope $P \subseteq \mathbb{R}^d$ with $d(K,P) \leq \varepsilon$.*

PROOF. Cover K by ε-balls:

$$K \subseteq \bigcup_{x \in K} B_\varepsilon(x)$$

Since K is compact we can find a finite subcover of K, thus there are $x_1, \dots, x_N \in K$ such that

$$K \subseteq \bigcup_{i=1}^{N} B_\varepsilon(x_i)$$

Define

$$P = \operatorname{conv}(x_1, \dots, x_N).$$

Then we have

$$P \subseteq K \subseteq \overline{K_\varepsilon}$$

and

$$K \subseteq \bigcup_{i=1}^{N} B_\varepsilon(x_i) \subseteq \operatorname{conv}(B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_N)) = P + \varepsilon B_d$$

which implies $\mathrm{d}(K,P) \leq \varepsilon$. □

Observe that we can always assume that

- P has rational vertex coordinates.
- P is a **simplicial polytope**.

Recall that a polytope $P \subseteq \mathbb{R}^d$ is simplicial if every proper face of P is a simplex. Equivalently P is simplicial if every supporting hyperplane H of P contains less then d vertices of P.

PROPOSITION 63. Let $P = \operatorname{conv}(v_1, \dots v_n) \subseteq \mathbb{R}^d$ be a d-polytope and $\varepsilon > 0$. Then there is a polytope $P' = \operatorname{conv}(v_1', \dots, v_n')$ with $\|v_i - v_i'\| \leq \varepsilon$ for all i and P' is simplicial.

PROOF. Exercise. □

The proof of Theorem 62 is highly nonconstructive and, in particular, does not give an idea of how large N gets with respect to K and ε. We will get back to this.

Lecture 8, May 6

2. Volumes of convex bodies

The notion of volume is fundamental in geometry. It is very intuitive but rather difficult to make rigorous. In a first attempt, we make use of concepts from real analysis to define the volume of a convex body as its Jordan measure. This is very pragmatic and it works. In a second attempt, we will try to develop the notion of volume from a somewhat axiomatic point-of-view.

2.1. Volume as Jordan measure. For a convex body $K \subseteq \mathbb{R}^d$, we can simply define

$$\operatorname{vol}_d(K) := \int_K 1 d\mu$$

For this, however, we need to know that convex bodies are Lebesque measurable.

A **box** $B \subseteq \mathbb{R}^d$ is a set of the form

$$B = \{x \in \mathbb{R}^d : a_i \leq x_i \leq b_i, i = 1, \dots, d\}.$$

for some $a_i \leq b_i$, $i = 1, \dots, d$. The volume of such a box B is $V(B) = \prod_{i=1}^{d}(b_i - a_i)$. A set $S \subset \mathbb{R}^d$ is called a **polybox** if

$$S = B_1 \cup B_2 \cup \dots \cup B_k$$

where $B_1, \dots, B_k$ are boxes with disjoint interiors. We write $\mathcal{B}_d$ for the collection of polyboxes in $\mathbb{R}^d$.

polybox_lattice LEMMA 64. If $A, B \in \mathcal{B}_d$ are polyboxes, then so is $A \cap B$ and $A \cup B$. Stronger, there are boxes $A_1, \ldots, A_k, B_1, \ldots, B_l, C_1, \ldots, C_m$ such that

$$\begin{aligned} A &= A_1 \cup \cdots A_k \cup C_1 \cup \cdots \cup C_m \\ B &= B_1 \cup \cdots B_l \cup C_1 \cup \cdots \cup C_m \\ A \cup B &= A_1 \cup \cdots A_k \cup B_1 \cup \cdots B_l \cup C_1 \cup \cdots \cup C_m \\ A \cap B &= C_1 \cup \cdots \cup C_m \end{aligned}$$

are representations of polyboxes.

PROOF. Homework! □

The **volume** of a polybox S is defined by $V(S) := \sum_{i=1}^{k} \mathrm{vol}_d(B_i)$ where $S = B_1 \cup \cdots \cup B_k$ is a representation as a polybox.

rop:polybox_vol PROPOSITION 65. The volume of a polybox is well-defined.

PROOF. Homework! □

The volume of polyboxes enjoys the following properties

lybox_vol_props PROPOSITION 66. Let $A, B \in \mathcal{B}_d$ polyboxes. Then the following properties hold:

i) **translation invariant**: $V(t + A) = V(A)$ for all $t \in \mathbb{R}^d$.
ii) **homogeneous**: $V(\lambda A) = \lambda^d V(A)$ for $\lambda \geq 0$.
iii) **monotone**: $V(A) \leq V(B)$ if $A \subseteq B$.
iv) **valuation**:

$$V(A \cup B) = V(A) + V(B) - V(A \cap B).$$

v) **simple** as a valution: $V(A) = 0$ if A is contained in a hyperplane.

PROOF. i), ii), and v) are certainly true if A is a box and follow for general polyboxes from the definition of volume. Lemma 64 implies iii) and iv). □

A set $S \subset \mathbb{R}^d$ is **Jordan measurable** if

$$\sup\{V(A) : A \in \mathcal{B}_d, A \subseteq S\} = \inf\{V(A) : A \in \mathcal{B}_d, S \subseteq A\}$$

Jordan measurable in particular implies Lebesgue measurable and $V(S) = \int_S d\mu$ for any Jordan measurable set.

THEOREM 67. *A convex body $K \subseteq \mathbb{R}^d$ is Jordan measurable. If K is contained in a hyperplane, then $V(K) = 0$.*

PROOF. We can assume that $0 \in \mathrm{relint}(K)$ and let $\varepsilon > 0$. Assume $0 \in \mathrm{int}\, K$ and pick $1 > \varepsilon > 0$. Then $\mathrm{d}((1-\varepsilon)K, K) = \delta > 0$ and for every point $p \in (1-\varepsilon)K$, we have $B_\delta(p) \subset K$. Hence, we can cover $(1-\varepsilon)K$ by small boxes $B \subseteq K$. Since $(1-\varepsilon)K$ is compact, we can assume that this cover S is finite and Lemma 64 S is a polybox. So we have

$$(1-\varepsilon)K \subseteq B \subseteq K.$$

and thus

$$B \subseteq K \subseteq \tfrac{1}{1-\varepsilon} B.$$

Proposition 66 then yields

$$V(B) \leq V(K) \leq \tfrac{1}{(1-\varepsilon)^d} V(B).$$

If K is contained in a hyperplane, then we can cover K by boxes of arbitrarily small volume. □

Of course, this definition is not quite satisfactory as volume computations involve a limit argument.

2.2. Volume from scratch. Can we develop from "first principles" without appealing to Jordan measurable sets? Is the volume uniquely defined by the natural properties of Proposition 66 together with the requirement $V([0,1]^d) = 1$?

The uniqueness is *not* a consequence of the last section: The Lebesque measure is unique with respect to all measurable sets *but* convex bodies/polytopes only form a small subset.

2.2.1. *Volume via equidissectability.* A promising approach towards volume is to define it through the valuation property.

DEFINITION 68. Let $P \subseteq \mathbb{R}^d$ be a d-polytope. A **dissection** of P is a collection of d-polytopes $P_1, \dots, P_k$ such that

$$P \;=\; P_1 \cup \cdots \cup P_k$$

with $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \varnothing$ for all $i \neq j$.

Two polytopes $P, Q \subseteq \mathbb{R}^d$ are **equidissectable** or **scissor congruent** if there are dissections

$$P \;=\; P_1 \cup \cdots \cup P_k \quad \text{and} \quad Q \;=\; Q_1 \cup \cdots \cup Q_k$$

such that P_i is congruent to Q_i for all $i = 1, \dots, k$.

If $P, Q \subseteq \mathbb{R}^d$ are equidissectable polytopes, then $V(P) = V(Q)$. The idea now is to define $V(P) = \lambda$ if P is scissor congruent to $Q = \lambda \cdot [0,1]^d$.

Hilbert's 3rd problem (ICM, 1900): Let $P, Q \subseteq \mathbb{R}^d$ be polytopes does $\mathrm{V}(P) = \mathrm{V}(Q)$ imply that P and Q are scrissor congruent?

This is clearly true in dimension $d = 1$ and not so hard in dimension $d = 2$. In dimension 3 it turns out that the answer is *no* with the following history.

- In 1844 Gauss expressed doubt.
- In 1897 Bricard gave a *wrong* proof that the problem is false.
- A real proof was given bx Max Dehn, a student of Hilbert. He defined what is now called the **Dehn invariant** $\mathrm{D}(\cdot)$ which has the property that $\mathrm{D}(P) = \mathrm{D}(Q)$ whenever $P, Q \subseteq \mathbb{R}^d$ are scissor congruent. For a regular tetrahedron $\Delta \subseteq \mathbb{R}^3$ with volume 1, Dehn showed $\mathrm{D}(\Delta) \neq 0$ but $\mathrm{D}([0,1]^3) = 0$.

It turns out that the Dehn invariant and the volume decides equidissectability: If $V(P) = V(Q)$ and $\mathrm{D}(P) = \mathrm{D}(Q)$ then P, Q are scissor congruent. This holds in dimension $d \leq 4$. Open in $\dim \geq 5$.

Hadwidger proved that in general dimension d, P, Q are equidissectable if and only if $\phi(P) = \phi(Q)$ for *every* rigid-motion invariant valuations ϕ.

We saw a very important valuation last semester.

THEOREM 69. *There is a unique valuation χ on closed convex sets—the **Euler characteristic**—such that $\chi(\varnothing) = 0$ and $\chi(K) = 1$ for $K \in \mathcal{K}_d$.*

Hadwiger also showed that the volume is the unique valuation $V : \mathcal{K}_d \to \mathbb{R}$ that is continuous in the Hausdorff metric and that satisfies the conditions of Proposition 66.

2.2.2. *An approach via simplices.* The route that we will take is that to

- define the volume on simplices and
- show that every polytope can be dissected into simplices.

A **k-simplex** $\Delta \subset \mathbb{R}^d$ is $\Delta = \mathrm{conv}(v_0, \dots, v_k)$ for some affinely independent points $v_0, \dots, v_k$. For a d-dimensional simplex Δ, we define

$$V(\Delta) \;:=\; \frac{1}{d!} \left| \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_0 & v_1 & \cdots & v_d \end{pmatrix} \right|$$

and we set $V(\Delta) = 0$ if $\dim \Delta < d$.

Properties of the determinant yield that $V(\cdot)$ is rigid motion invariant. If P is a polytope with dissection $P = \Delta_1 \cup \cdots \cup \Delta_m$ for simplices Δ_i, we define

$$V(P) := \sum_i V(\Delta_i)$$

The obvious question is whether this is well-defined, that is, it should not depend on the dissection.

For now, let us write $\mathcal{S}_d = \{\Delta \subset \mathbb{R}^d : \Delta \text{ simplex}\}$. You will show in the exercises, that

PROPOSITION 70. V is a valuation on $\mathcal{S}_d$. That is,

$$V(\Delta \cup \Delta') \;=\; V(\Delta) + V(\Delta') - V(\Delta \cap \Delta').$$

whenever $\Delta, \Delta', \Delta \cup \Delta', \Delta \cap \Delta' \in \mathcal{S}_d$.

THEOREM 71. *Every valuation on simplices can be extended to a valuation on polytopes.*

Let us now check that $V([0,1]) = 1$. For this we do the following. We call a point $p \in [0,1]^d$ a *general* point if $0 \neq p_i \neq p_j \neq 1$ for $i \neq j$. Consider the polyhedron

$$S \;=\; \{p \in [0,1]^d : 0 < p_1 < p_2 < \cdots < p_d < 1\}$$

This is the interior of a d-dimensional polytope and we check that the vertices of the closure of S are

$$0, e_1, e_1 + e_2, \ldots, e_1 + \cdots + e_d$$

for $0 \leq j \leq d$. Hence $\Delta := \bar{S}$ is a simplex of volume $\frac{1}{d!}$.

For any general point $p \in [0,1]^d$, there is a unique permutation σ such that

$$0 < p_{\sigma(1)} < p_{\sigma(2)} < \cdots < p_{\sigma(d)} < 1$$

The permutation induces a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ with

$$T_\sigma(x_1, \ldots, x_d) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(d)}).$$

This is a (improper?) rigid motion and we see

$$[0,1]^d \;=\; \bigcup_\sigma T_\sigma(\Delta).$$

Observe that Δ and $T_\sigma(\Delta)$ have disjoint interiors and hence

$$V([0,1]^d) = \sum_\sigma V(\Delta) = \frac{d!}{d!}$$

Lecture 9, May 12

2.3. Barycentric subdivisions and volume formulas I. We now have the notion of volume that we can explicitly compute for simplices but in order to compute it for a general polytope P, we need a dissection of P.

Let $P \subset \mathbb{R}^d$ be a fixed, full-dimensional polytope. For every nonempty face $F \subseteq P$, pick a point $b_F \in \operatorname{relint}(F)$. A **flag of faces** is a sequence

$$\mathcal{F} \;=\; \{F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k\}$$

such that $\dim F_i < \dim F_{i+1}$ for all $i = 0, \ldots, k-1$. Then **length** of a flag is k. The flag is **complete** if $k = d$ and hence $\dim F_i = i$. For a flag $\mathcal{F}$ define

$$\Delta(\mathcal{F}) \;:=\; \operatorname{conv}(b_{F_i} : i = 1, \ldots, d).$$

PROPOSITION 72. Let $\mathcal{F}$ be a flag of length k, then $\Delta(\mathcal{F})$ is a k-simplex.

Proof. We prove the result by induction on k. If $k = 0$, then $\mathcal{F} = \{F\}$ and $\Delta(\mathcal{F})$ is a 0-simplex. Assume that the claim is true for all flags of length $< k$ and let

$$\mathcal{F} \;=\; \{F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1} \subseteq F_k\}.$$

be a flag of length k. Then

$$\mathcal{F}' \;=\; \{F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1}\}$$

and by induction $\Delta(\mathcal{F}')$ is a $(k-1)$-simplex and $b_{F_0}, \ldots, b_{F_{k-1}}$ affinely independent. Observe that $b_{F_i} \in F_{k-1} \subset \partial F_k$. Hence, b_{F_k} is not contained in the affine span of $\Delta(\mathcal{F}')$ and the points $b_{F_0}, \ldots, b_{F_k}$ affinely independent. Hence $\Delta(\mathcal{F})$ is a k-simplex. □

Theorem 73. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be the set of all complete flags of P, then*

$$P = \Delta(\mathcal{F}_1) \cup \cdots \cup \Delta(\mathcal{F}_m)$$

is a dissection of P into simplices.

For reasons that will become apparent soon we call this the **barycentric subdivision** of P.

Proof. We already proved that every $\Delta(\mathcal{F}_i)$ is a simplex. So we only have to show that We have to prove:

(i) The simplices $\Delta(\mathcal{F}_i)$ cover P.
(ii) For any two complete flags $\mathcal{F}_i, \mathcal{F}_j$ with $i \neq j$, we have

$$\operatorname{int}(\Delta(\mathcal{F}_i)) \cap \operatorname{int}(\Delta(\mathcal{F}_j)) \;=\; \varnothing.$$

We prove both properties by induction on $d = \dim P$. This is certainly true for $d = 1$ (**Picture!**).
Let $p \in P$ be an arbitrary point. If $p \in \partial P$, then $p \in F$ for some facet F. By induction, there is a complete flag

$$\mathcal{F}' = \{F_0 \subset F_1 \subset \cdots \subset F_{d-1} = F\}$$

such that $p \in \Delta(\mathcal{F}')$.
If $p \in \operatorname{int}(P)$, then the ray $b_P + \mathbb{R}_{\geq 0}(p - b_P)$ meets the boundary of P in a unique point r. By induction $r \in \Delta(\mathcal{F}')$ and extending the flag by $F_d = P$, proves (1).
As for (2), assume that $p \in \operatorname{int}(\Delta(\mathcal{F}_i)) \cap \operatorname{int}(\Delta(\mathcal{F}_j))$. and let again r be the unique point in $\partial P \cap (b_P + \mathbb{R}_{\geq 0}(p - b_P))$. Since $\Delta(\mathcal{F}_i)$ is a simplex, it follows that if r is in the relative interior of a facet of $\Delta(\mathcal{F}_i)$. These facets of $\Delta(\mathcal{F}_i)$ and $\Delta(\mathcal{F}_j)$ hence meet in a relative interior point in the boundary of P. However, by induction (2) holds. A contradiction. □

Hence, the volume of a polytope P is

$$V(P) \;=\; \sum_{\mathcal{F} \text{ full flag}} V(\Delta(\mathcal{F})).$$

We can use the barycentric subdivision to get a very general formula of the volume of any polytope.
Let $Q \subset \mathbb{R}^d$ be a $(d-1)$-polytope and $v \notin \operatorname{aff}(Q)$. Then $P = \operatorname{conv}(v \cup Q)$ is called a **pyramid** over Q with apex v. In particular if $P = \operatorname{conv}(v_0, \ldots, v_k)$ is a k-simplex, then P is a pyramid over the $(k-1)$-simplex $\operatorname{conv}(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k)$ for $0 \leq i \leq k$ with apex v_i.
If P is a full-dimensional pyramid, then Q is a facet with supporting hyperplane $H = \{x : a^t x = b\}$. We call $h = \frac{b - a^t v}{\|a\|}$ the **height** of the pyramid.

Lemma 74. Let P be a pyramid with base Q and height h. Then

$$\operatorname{vol}_d(P) \;=\; \frac{1}{d} \cdot h \cdot \operatorname{vol}_{d-1}(Q).$$

Here $V(Q)$ is the volume of Q restricted to $\operatorname{aff}(Q) = \mathbb{R}^{d-1}$.

PROOF. Let us first consider the case that $P = \operatorname{conv}(v_0, \dots, v_d)$ is a d-simplex and $Q = \operatorname{conv}(v_1, \dots, v_d)$. By an appropriate rigid motion, we can assume that the supporting hyperplane H of P that contains B is given by $H = \{x \in \mathbb{R}^d : x_1 = 0\}$. Thus

$$v_0 \;=\; \begin{pmatrix} -h \\ v_0' \end{pmatrix} \text{ and } v_i \;=\; \begin{pmatrix} 0 \\ v_i' \end{pmatrix} \text{ for } i = 1, \dots, d.$$

Then

$$\operatorname{vol}_d(P) \;=\; \frac{1}{d!}\left|\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ -h & 0 & \dots & 0 \\ v_0' & v_1' & \dots & v_d' \end{pmatrix}\right| \;=\; \frac{h}{d}\frac{1}{d!}\left|\det\begin{pmatrix} 1 & \dots & 1 \\ v_1' & \dots & v_d' \end{pmatrix}\right| \;=\; \frac{1}{d}\cdot h \cdot \operatorname{vol}_d(Q).$$

For a general pyramid P and base Q, let $Q_1, \dots, Q_M$ be the dissection of Q into simplices coming, for example, from the barycentric subdivision. Then $P_i := \operatorname{conv}(v \cup Q_i)$ for $i = 1, \dots, M$ yields a dissection of P into simplices P_i. Observe that all simplices have the same height h. Hence, we compute

$$V(P) \;=\; \sum_i V(P_i) = \sum_i \frac{1}{d!} h V(Q_i) = \frac{1}{d} h \sum_i V(Q_i) = \frac{1}{d} h V(Q). \qquad \square$$

The following gives a very useful formula for the volume of a polytope. For a polytope $P \subset \mathbb{R}^d$ and a vector $c \in \mathbb{R}^d$, we write

$$P^c \;:=\; P \cap \{x : c^t x = h_P(c)\}$$

THEOREM 75. *Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with unit facet normals $a_1, \dots, a_m$. Then*

$$\operatorname{vol}_d(P) \;=\; \frac{1}{d} \sum_{i=1}^m h_P(a_i) \operatorname{vol}_{d-1}(P^{a_i}).$$

PROOF. Let $q \in \operatorname{int}(P)$ be an arbirary point. The facets of P are

$$F_i \;=\; P \cap \{x : a_i^t x = b_i\}$$

with $b_i = h_P(a_i)$. Define $P_i := \operatorname{conv}(\{q\} \cup F_i)$. It is easy to see that

$$P \;=\; P_1 \cup \dots \cup P_m$$

is a dissection of P into pyramids over the facet F_i with apex q. The height of the pyramid F_i is

$$\frac{b_i - a_i^t q}{\|a_i\|} \;=\; h_P(a_i) - a_i^t q.$$

Hence, we verify

$$V(P) \;=\; \sum_{i=1}^m V(P_i) \;=\; \frac{1}{d} \sum_{i=1}^m h_P(a_i) V(F_i) - \left(\sum_{i=1}^m V(F_i) a_i\right)^t q.$$

The following lemma completes the proof of the theorem. $\square$

lem:nopressure

LEMMA 76. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with unit facet normals $a_1, \dots, a_m$. Then

$$\sum_{i=1}^m V(P^{a_i}) a_i \;=\; 0.$$

PROOF. Let $P = \{x \in \mathbb{R}^d : a_i^t x \le b_i \text{ for } i = 1, \dots, m\}$. Pick z a unit vector and let $\pi_z : \mathbb{R}^d \to z^\perp = \{x : z^t x = 0\}$ be the corresponding orthogonal projection. Consider the polytope $Q = \pi_z(P)$. For every point $q \in Q$, the fiber

$$\pi_z^{-1}(q) \cap P \;=\; \{q + \lambda z : a_i^t q + (a_i^t z)\lambda \le b_i \text{ for } i = 1, \dots, m\}$$

is a polytope of dimension ≤ 1. Define

$$I^+ \;:=\; \{1 \le i \le m : a_i^t z > 0\}$$
$$I^- \;:=\; \{1 \le i \le m : a_i^t z < 0\}.$$

1.Claim: Two dissections of Q are given by

$$\bigcup_{i\in I^+} \pi_z(F_i) \;=\; Q \;=\; \bigcup_{i\in I^-} \pi_z(F_i)$$

To check that both unions cover Q it is sufficient to check that if $\pi_z^{-1}(q) = [q^-, q^+]$, then $q^+ \in F_i$ for some $i \in I^+$ and $q^- \in F_j$ for some $j \in I^-$. That the individual pieces do not meet in the relative interior follows from

2.Claim: $\pi_z(F_i)$ is affinely isomorphic to F_i for $i \in I^+ \cup I^-$.

The affine hull of F_i is given by $\operatorname{aff}(F_i) = \{x : a_i^t x = b_i\}$. The map $s : z^\perp \to \operatorname{aff}(F_i)$ given by

$$s_i(p) \;:=\; p + \frac{b_i - a_i^t p}{a_i^t z} z$$

is an inverse of π_z restricted to $\operatorname{aff}(F_i)$. Hence, if $q \in \operatorname{relint}(\pi_z(F_i)) \cap \operatorname{relint}(\pi_z(F_j))$ with $i, j \in I^+$, then $\pi_z^{-1}(q)$ meets the relative interiors of F_i and F_j. But these are disjoint since both are faces of P. The same arguments works for I^-.

3.Claim: $V_{d-1}(\pi_z(F_i)) = |a_i^t z| V_{d-1}(F_i)$.

This is best checked for the case F_i being a $(d-1)$-simplex where we have explicit (determinant) formulas. But then this is true by using a dissection of F_i into simplices.

Finally, we compute

$$\sum_{i\in I^+} z^t a_i V(F_i) \;=\; V(Q) \;=\; -\sum_{i\in I^-} -z^t a_i V(F_i).$$

Together this yields

$$\left(\sum_{i=1}^{m} V(F_i) a_i\right)^t z \;=\; 0$$

Since z was chosen arbirarily, this proves the claim. □

Physical interpretation!

This result implies the easy direction of the following deep representation theorem for polytopes that we will study later.

THEOREM 77 (Minkowski's existence/uniqueness theorem). *Let $a_1, \dots, m_m$ be unit vectors spanning $\mathbb{R}^d$ and $\alpha_1, \dots, \alpha_m > 0$. There is polytope P with facet directions a_i and corresponding facet volumes α_i if and only if*

$$\sum_i \alpha_i a_i \;=\; 0.$$

The polytope P is unique up to translation.

2.4. Beneath-beyond and placing triangulations. **Not added yet. Please see your own notes.**