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Discrete Geometry I

Homework # 13 — due February 13th

Exercise 1. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be an arrangement of n distinct, linear hyperplanes $H_i = \{x \in \mathbb{R}^d : z_i^t x = 0\}$. Denote by $r(\mathcal{H})$ the number of regions of $\mathcal{H}$.

i) Show that $r(\mathcal{H}) \;=\; r(\mathcal{H} \setminus H_i) + r(\mathcal{H}|_{H_i})$ and verify that $r(\mathcal{H}) \leq 2^n$.

ii) Show that the inequality is strict for $n > d$.
[Hint: You might need Radon's lemma.]

iii) Show that for $d = 3$ we have $r(\mathcal{H}) \leq n(n-1) + 2$ and that this bound is sharp.

iv) (Bonus) Show that

$$r(\mathcal{H}) \;=\; (-1)^d \sum_{J \subseteq [n]} (-1)^{|J| - \dim H_J} \;=\; \sum_{J \subseteq [n]} (-1)^{|J| - \mathrm{rank}\{z_i \,:\, i \in J\}}$$

where $H_J = \cap_{i \in J} H_i$.
[Hint: Verify that the right-hand side satisfies the same recursion as in i)]

(10+3 points)

Exercise 2. Let $P \subset \mathbb{R}^2$ be the hexagon with vertices

$$V(P) \;=\; \left\{ \pm\binom{2}{0}, \pm\binom{0}{2}, \pm\binom{2}{2} \right\}.$$

Determine $\mathcal{T}(P)$, its dimension, its vertices, and its combinatorial structure. Describe the summands corresponding to faces of $\mathcal{T}(P)$.

(13 points)

Exercise 3.

i) Describe the indecomposable summands of a regular n-gon.

ii) Show that the join of two polytopes is always indecomposable.

iii) Find two combinatorially equivalent polytopes in $\mathbb{R}^3$, one of which is decomposable and the other is not.

(13 points)