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WS2012/13

Discrete Geometry I

Homework # 12 — due February 6th

Exercise 1. i) Without using Gale diagrams, show that *every simplicial* d-polytope with $d+3$ vertices has at least $\binom{d+3}{2} - 3$ edges.

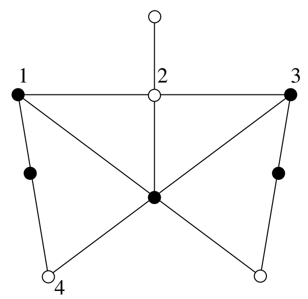
ii) Using Gale diagrams show that, for $d \geq 3$, *every* d-polytope with $d+3$ vertices has at least $\binom{d+3}{2} - 4$ edges.

iii) Describe the (unique!) d-polytope with $f_0 = d+3$ and $f_1 = \binom{d+3}{2} - 4$.

(10 points)

Exercise 2. i) Let $F = \text{conv}(v_1, \ldots, v_k)$ be a face of a convex d-polytope $P = \text{conv}(v_1, \ldots, v_n)$ in $\mathbb{R}^d$. Describe how to obtain the *affine* Gale diagram of F from the *affine* Gale diagram of P.

ii) We have seen that the diagram below is an affine Gale diagram of a polytope. Show that the vertices $1, 2, 3$ form a cofacet and describe the combinatorial type of the corresponding facet.



iii) How many vertices does the vertex figure of the vertex 4 have? Draw its affine Gale diagram.

(10 points)

(continued on backside)

Exercise 3.

i) Show that the normal fan of a polytope coincides with the face fan of its polar dual: $\mathcal{N}(P) = \mathcal{F}(P^\triangle)$.

ii) Describe the combinatorial structure of the Minkowski sum $P + P^\triangle$, where P is a 3-simplex containing the origin in its interior.

iii) Let $P_1, P_2, \ldots, P_k \subset \mathbb{R}^2$ be polygons. Show that

$$f_0(P_1 + P_2 + \cdots + P_k) \;\leq\; f_0(P_1) + f_0(P_2) + \cdots + f_0(P_k)$$

iv) (Bonus) What is the maximum number of vertices the Minkowski sum of two 3-simplices can have?

(10+3 points)

Exercise 4.

i) Show that the diameter of a zonotope $Z = \sum_{i=1}^{n}[-z_i, z_i]$ (where vectors z_i are pairwise non-collinear) equals n. Which pairs of points realize the maximum distance?

ii) Let $G = ([n], E)$ be a simple (no parallel edges, no loops), connected graph and define the graphical zonotope

$$\mathcal{Z}(G) \;:=\; \sum_{ij \in E} [e_i - e_j, e_j - e_i]$$

Show that the vertices are in bijection to acyclic orientations of G, i.e. orientations of the edges of G without directed cycles.

iii) Let P be a d-polytope such that each of its 2-dimensional projections is a zonotope. Is then P itself necessarily a zonotope?

iv) (Bonus) Show that P is a zonotope if every projection of P to $\mathbb{R}^3$ is a zonotope.

(10+3 points)