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Discrete Geometry I

Homework # 11 — due January 30th

Exercise 1. Let $V = (v_1, v_2, \ldots, v_n) \subset \mathbb{R}^{d\times n}$ be a (affine) point(!) configuration and $G = (g_1, \ldots, g_n) \in \mathbb{R}^{(n-d-1)\times n}$ a linear Gale transform for V. A projective transformation

$$T(x) \;=\; \frac{Ax+b}{c^t x + \delta}$$

(with A non-singular) is *admissible* for V if $c^t v_i + \delta > 0$ for all $i = 1, \ldots, n$.

i) Show that $G' = (\lambda_1 g_1, \lambda_2 g_2, \ldots, \lambda_n g_n)$ is a Gale transform of $T(V)$ for suitable $\lambda_1, \ldots, \lambda_n$.

ii) Infer that two polytopes are projectively equivalent if and only if they have the same affine Gale transform.

(10 points)

Exercise 2. A matrix $\bar{G} = (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_n) \in \mathbb{R}^{(n-d-2)\times n}$ together with $\sigma \in \{-1, +1\}^n$ specifies an affine Gale diagram ($\bar{g}_i$ is positive, if $\sigma_i > 0$).

i) Show that $G' \subseteq G$ is a coface if and only if

$$\operatorname{conv}(g_i \in G' : \sigma_i > 0) \;\cap\; \operatorname{conv}(g_i \in G' : \sigma_i < 0) \;\neq\; \emptyset$$

ii) Show that G is an affine Gale diagram of a k-neighborly polytope
$\Leftrightarrow$ i) holds for every $G' \subseteq G$ with $|G'| = n-k$
$\Leftrightarrow$ for every affine hyperplane H

$$|G^+ \setminus H^-| + |G^- \setminus H^+| \;\geq\; k+1.$$

iii) For $t_1 < t_2 < \cdots < t_n$ with $n > d > 2$, define

$$\bar{g}_i \;:=\; (t_i, t_i^2, \ldots, t_i^{n-d-2})^t \quad \text{and} \quad \sigma_i = (-1)^i$$

for $i = 1, 2, \ldots, n$. Show that this gives the affine Gale transform of a d-dimensional $\lfloor \frac{d}{2} \rfloor$-neighborly polytope on n vertices.

(10 points)

Exercise 3. A set $B = \{b_1, b_2, \ldots, b_k\} \subset \mathbb{R}^d$ is *minimally positively spanning* if B is positively spanning but every proper subset is not.

i) Show that $|B| \geq d+1$.

ii) Show that $|B| \leq 2d$.
For that, phrase this in terms of a Gale transform of B.

iii) Characterize those B for which $|B| = 2d$.

(10 points)

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Exercise 4. Let $G = (g_1, \dots, g_n) \in \mathbb{R}^{(n-d-1)\times n}$ be the Gale transform of a d-polytope with n vertices.

i) Show that $G' = \left(\frac{1}{2}g_1, \frac{1}{2}g_1, g_2, \dots, g_n\right)$ is the Gale transform of a polytope and determine its combinatorics.

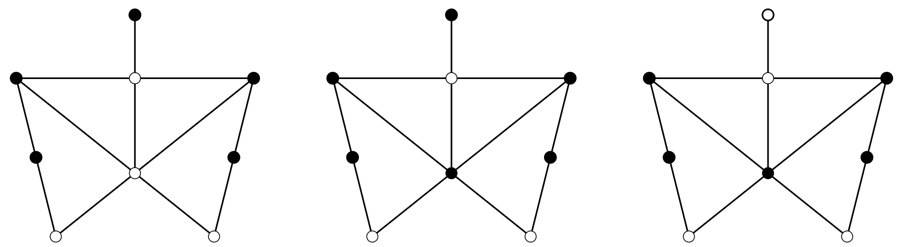
ii) Let $G_1 = (g_1, \dots, g_{n_1}) \in \mathbb{R}^{(n_1-d_1-1)\times n_1}$ and $G_2 = (h_1, \dots, h_{n_2}) \in \mathbb{R}^{(n_2-d_2-1)\times n_2}$ be the Gale transforms of a d_1-polytope P_1 and a d_2-polytope P_2, respectively. Describe the polytope for which

$$G = \begin{pmatrix} g_1 & \cdots & g_{n_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & h_1 & \cdots & h_{n_2} \end{pmatrix}$$

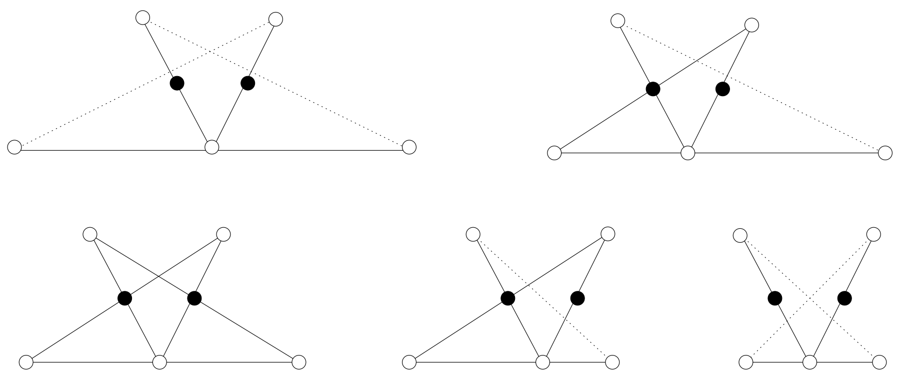
is a Gale transform.

(10 points)

Exercise 5. i) Which of the following point configurations are affine Gale diagrams of polytopes? Why, or why not?



ii) Consider the following diagrams. Verify that they are affine Gale diagrams and construct the associated polytopes.



(10 points)