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Discrete Geometry I

Homework # 10 — due January 23rd

Exercise 1. Let $P \subset \mathbb{R}^2$ be a convex polygon. A *triangulation* of P is a geometric simplicial complex Δ such that

$$\bigcup_{F \in \Delta} F \;=\; P$$

i) Show that Δ is pure and strongly connected.

ii) Show that Δ is shellable.

iii) (Bonus) Show that Δ is *extendably* shellable, that is, the first facet can be chosen arbitrarily.

(10+3 points)

Exercise 2. Let $0 \leq k < n-1$. The k-*skeleton* of the $(n-1)$-simplex is

$$\mathrm{skel}_k(\Delta_{n-1}) \;:=\; \{F \in \mathcal{L}(\Delta_{n-1}) : \dim F \leq k\}$$

This is a pure k-dimensional simplicial complex Δ.

i) Show that the reverse-lexicographic order of the facets of Δ gives a shelling order.

ii) What is the h-vector of Δ?

(10 points)

Exercise 3. Let $\tilde{A} \subseteq [n]$ be a multisubset of size k, that is, $\tilde{A} : [n] \to \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{n} \tilde{A}(i) = k$.

i) Show that the map

$$\begin{array}{rccc} \phi : & \left(\!\!\binom{n}{k}\!\!\right) & \to & \binom{n+k-1}{k} \\ & \{a_1 \leq a_2 \leq \cdots \leq a_k\} & \mapsto & \{a_1 < a_2 + 1 < \cdots < a_k + k - 1\} \end{array}$$

is a bijection between k-multisubsets of $[n]$ and k-subsets of $[n+k-1]$.

ii) Let $\mathcal{F}$ be the collection of k-multisubsets reverse-lexicographically smaller than $\tilde{A}$. Verify that $\partial\mathcal{F}$ is compressed.

iii) Let $B \subseteq [n]$ be a subset ($B : [n] \to \{0,1\}$). Give a formula for

$$\left|2^{[n]}_{\prec_{\mathrm{rl}} B}\right| \;=\; |\{A \subseteq [n] : A \prec_{\mathrm{rl}} B\}|$$

What is it for the multisubsets of $\tilde{A}$ of size at most $|\tilde{A}|$?

(10 points)

(continued on backside)

Exercise 4. Let $f = (23, 47, 52, 38, 12)$.

i) Is f the f-vector $(f_0, f_1, \ldots, f_4)$ of a 4-dimensional simplicial complex?

ii) Is f the f-vector of a shellable complex?

iii) Is f the f-vector of a simplicial 5-polytope?

(10 points)