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Discrete Geometry I

Homework # 9 — due January 16th

For a finite set V, a *simplicial complex* with vertices in V is $\emptyset \neq \Delta \subseteq 2^V$ such that $G \subseteq F \in \Delta$ implies $G \in \Delta$.

Exercise 1. For two (abstract) simplicial complexes $\Delta_1 \subseteq 2^{V_1}$, $\Delta_2 \subseteq 2^{V_2}$ with $V_1 \cap V_2 = \emptyset$, define their *join* as

$$\Delta_1 * \Delta_2 \ := \ \{F_1 \cup F_2 \mid F_1 \in \Delta_1, F_2 \in \Delta_2\} \ \subseteq \ 2^{V_1 \cup V_2}.$$

i) Show that $\Delta_1 * \Delta_2$ is a simplicial complex and that $\dim(\Delta_1 * \Delta_2) = \dim \Delta_1 + \dim \Delta_2$.

ii) Show that if Δ_1 and Δ_2 are pure and strongly connected, then so is their join.

iii) Show that

$$\mathrm{lk}_{\Delta_1 * \Delta_2}(F_1 \uplus F_2) = \mathrm{lk}_{\Delta_1}(F_1) * \mathrm{lk}_{\Delta_2}(F_2)$$

and infer that the join of Eulerian manifolds is again an Eulerian manifold.

(10 points)

Exercise 2. Let P be a d-polytope. A *flag* of faces is a chain $\emptyset \neq F_0 \subset F_1 \subset \cdots \subset F_k$ of non-empty faces $F_1, \dots, F_k \in \mathcal{L}(P)$.

i) Show that the collection of all flags forms a pure and strongly connected d-dimensional simplicial complex $\mathrm{sd}(P) \subseteq 2^{\mathcal{L}(P)}$.

For every non-empty face F, let p_F be a point in the relative interior of F.

ii) Show that for a chain $\mathcal{F} = \{F_0 \subset F_1 \subset \cdots \subset F_k\} \in \mathrm{sd}(P)$

$$\widehat{\mathcal{F}} \ := \ \mathrm{conv}\{p_{F_0}, p_{F_1}, \dots, p_{F_k}\}$$

is a simplex of dimension $k = \dim \mathcal{F}$.

iii) Show that for every point $p \in P$ there is a unique flag $\mathcal{F} \in \mathrm{sd}(P)$ such that $p \in \mathrm{relint}\, \widehat{\mathcal{F}}$.

[Hint: Try induction on the dimension and consider faces.]

iv) Infer from iii) that $\{\widehat{\mathcal{F}} : \mathcal{F} \in \mathrm{sd}(P)\}$ is a geometric simplicial complex.

Remark: this simplicial complex is called a *barycentric subdivision* of P.

(10 points)

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Exercise 3. Let Δ be a finite simplicial complex.

i) For a nonempty face $F \in \Delta$ show that

$$\mathrm{lk}_\Delta(F) \;=\; \{G \setminus F : F \subseteq G, G \in \Delta\}$$

ii) Show that if Δ is an Eulerian manifold, then Δ is a pseudo-manifold. Is Δ necessarily strongly connected?
[Hint: What is the link of a face of dimension $\dim \Delta - 1$?]

Let $\Delta = \mathcal{B}(P)$ be the boundary complex of a simplicial d-polytope.

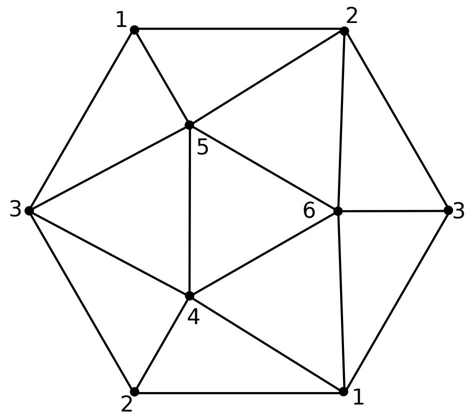
iii) For a non-empty face $F \in \Delta$, show that

$$\tilde{\chi}(\mathrm{lk}_\Delta(F)) \;=\; (-1)^{d-1-\dim F}\chi(P/F)$$

where P/F is the face figure of $F \subset P$.
[Hint: Consider the relation of $\mathrm{lk}_\Delta(F)$ and $\mathcal{L}(P/F)$.]

(10 points)

Exercise 4. Consider the following 2-dimensional simplicial complex Δ on 6(!) vertices. (Mind the identifications on the boundary!).



i) Compute $h(\Delta)$.

ii) Is Δ partitionable?

iii) Is Δ shellable?

(10 points)

Exercise 5. (Bonus) Show that every d-dimensional simplicial complex can be embedded in $\mathbb{R}^{2d+2}$. (Hint: embed it in the boundary of a suitable simplicial $(2d+2)$-polytope.)

(3 points)