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WS2012/13

Discrete Geometry I

Ho-Ho-Homework # $8\frac{1}{2}$ — due January 9th

These are all **bonus** problems. You may submit solutions to as many problems as you want.

Exercise 1. Find 4-polytopes with the following f-vectors:

i) $(10, 30, 40, 20)$;

ii) $(10, 26, 27, 11)$;

iii) $(10, 30, 30, 10)$

(Hint: Operations such as stacking, join etc. can help you in the first two cases.)

(3+3+6 points)

Exercise 2. Consider the following 4-polytopes:

$$\begin{aligned} P_1 &:= \{x \in [0,1]^5 \mid x_1 + x_2 + x_3 + x_4 + x_5 = 2\}, \\ P_2 &:= \{x \in [0,1]^4 \mid 1 \le x_1 + x_2 + x_3 + x_4 \le 2\}, \text{ and} \\ P_3 &:= \text{conv}(\Delta_4 \cup \Delta_4^{\triangle}), \end{aligned}$$

where Δ_4 is a 4-simplex that is edge-tangent to the unit sphere.

i) Show that P_1 is affinely equivalent to P_2.

ii) Show that P_3 is combinatorially equivalent to P_1 and P_2.

iii) Compute the f-vector of P_i. (By hand or using `polymake`.)

iv) Describe the combinatorial types of facets of P_i.

(3+6+3+6 points)

Exercise 3. Let P be a centrally-symmetric d-polytope: $P = -P$. Show that the total number of its non-empty faces is bounded from below as

$$f_P(1) := f_0 + f_1 + \ldots + f_d \ge 3^d$$

i) for $d = 2$ (trivial);

ii) for $d = 3$ (interesting);

iii) for $d = 4$ (challenging!);

iv) for $d \ge 5$ (open!!).

(3+9+27+81 points)

(continued on backside)

Exercise 4. i) Write down the Dehn-Sommerville equations for *simplicial* 5-polytopes.

ii) Compute the f-vector of the cyclic polytope $C_5(7)$. (Hint: you know not only the number of vertices, but also the number of edges of this polytope.)

iii) Does there exist a simplicial 5-polytope with 7 vertices and 13 facets?

(3+3+3 points)

Exercise 5. Let P be a simple 4-dimensional polytope and consider

$$K(P) := \sum_{F \subset P \text{ facet}} (f_0(F) - f_1(F) + f_2(F))$$

i) Show that $K(P) = 2f_3(P)$.
[Hint: What is the value of every summand?]

ii) Expand the sum and interpret each of the 4 sums individually to show that $K(P) = 4f_0(P) - 3f_1(P) + 2f_2(P)$.

iii) Together this gives a linear relation on the f-vector. How is it implied by the Dehn-Sommerville equations?

(3+3+3 points)