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Discrete Geometry I

Homework # 8 — due December 19th

Exercise 1. i) Show that if P is a simple and simplicial d-polytope for $d \geq 3$, then P is a simplex.

ii) Show that there is no 4-polytope with $f_0 = 6$ and $f_1 = 12$. (Hint: Try to infer the f-vector from these data.)

A graph is called *k-regular* if all of its vertices have degree k. For example, the graph of a simple d-polytope is d-regular.

iii) Give an example of a 4-connected 4-regular graph which is not the graph of a simple 4-polytope. (Hint: 6 vertices suffice.)

(10 points)

Exercise 2. i) For $\dim P = 4$ and $f_0 = 6$, describe all possible values of f_1 and give examples of 4-polytopes that realize these values.

ii) Construct a 4-polytope with 8 vertices and 16 edges.

iii) (Bonus) Show that there is no 4-polytope with 7 vertices and 14 edges.

(10+3 points)

Exercise 3. Let $P \subset \mathbb{R}^d$ be a d-polytope with facets $F_1, \ldots, F_m$ and correspoding supporting hyperplanes $H_i = \{x : a_i^t x = b_i\}$. A point $q \in \mathbb{R}^d$ is *beneath* F_i if $a_i^t q < b_i$ and *beyond* F_i if $a_i^t q > b_i$. Let $k \in [m]$ be fixed.

i) Show that there is a point $q_k \in \mathbb{R}^d$ such that q_k is beyond F_k and beneath F_i for all $i \neq k$. (Hint: Start from a well chosen point in F_k)

The operation of *stacking a vertex* onto the facet F_k of P is the polytope

$$\mathrm{stack}(P, F_k) \;=\; \mathrm{conv}(P \cup \{q_k\})$$

where q_k is as defined above.

ii) Show that 'stacking a facet' is dual to 'truncating a vertex', i.e.

$$\mathrm{trunc}(P, v)^{\triangle} \;\cong\; \mathrm{stack}(P^{\triangle}, v^{\diamondsuit})$$

In particular, the combinatorial type of $\mathrm{stack}(P, F_k)$ is independent of q_k. [Hint: Put P into the 'right position' and use polarity.]

iii) A d-dimensional *stacked polytope* on n vertices is the $(n-d-1)$-fold stacking of a d-simplex. Show that the f-vector is independent of the stacking order. Give an example of two stacked 3-polytopes on 7 vertices that are combinatorially distinct.

iv) (Bonus) Can you find a criterion (in dimension 3, say) when two different stacking orders of a d-simplex give combinatorially isomorphic polytopes?

(10+3 points)

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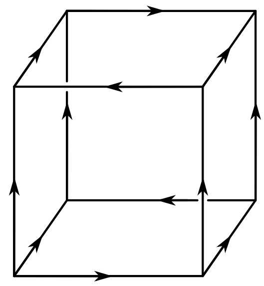
Exercise 4. Consider the following acyclic orientation of the graph of the 3-cube.



i) Show that the orientation is *good*. (Hint: You only have to compute a single number.)

ii) Show that this orientation is not induced by a linear function $\ell(x)$ on $C_3 = [-1,1]^3$.

iii) (Bonus) Is there a 3-polytope $P \subset \mathbb{R}^3$ and a linear function $\ell(x)$ that induces exactly this orientation?

(10+3 points)