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WS2012

Discrete Geometry I

Homework # 7 — due December 12th

Exercise 1. Let $P \subset \mathbb{R}^d$ be a simple polytope and $\ell(x)$ a linear function in general position with respect to P. The h-vector $h(P) = (h_0, h_1, \ldots, h_d)$ of P can be computed using ℓ: h_i is the number of vertices of P with exactly i neighbors with larger value.

i) By choosing a good ℓ show that $h_i(C_d) = \binom{d}{i}$ for $0 \leq i \leq d$.

ii) By choosing a good ℓ show that $h_i(\Delta_d) = 1$ for all i.

iii) Give a geometric interpretation for $h_1(P) = f_{d-1} - d$.

(10 points)

Exercise 2. i) Check that the Dehn-Sommerville equations for $d = 4$ are equivalent to $f_0(P) - f_1(P) + f_2(P) - f_3(P) = 0$ and $d\, f_0(P) = 2f_1(P)$.

ii) For $d = 5$, find a linear relation that follows from the Dehn-Sommerville equations but is independent of the Euler relation and $2f_1(P) = 5f_0(P)$.

(10 points)

Exercise 3. Let Δ_d be the d-simplex. Let H_k be a hyperplane such that $V(\Delta_d) \cap H_k = \emptyset$ and $|V(\Delta_d) \cap H_k^-| = k + 1$ with $0 \leq k \leq d$.

i) Show that $T_{d,k} := \Delta_d \cap H^-$ is a simple polytope.

ii) Determine the f- and h-polynomial of $T_{d,k}$.

iii) (Bonus) Find the dimension of the affine hull of the f-vectors of $T_{d,i}$.

(10+3 points)

Exercise 4. For $d \geq 1$ and $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$ let

$$Q_{d,i} \ := \ \mathrm{prism}^{(d-i)}(\Delta_i) \ = \ C_{d-i} \times \Delta_i$$

be the $(d - i)$-fold prism of an i-simplex.

i) Determine the f-polynomial $f_{d,i}(t)$ and h-polynomial $h_{d,i}(t)$ of $Q_{d,i}$ for all i.

ii) Deduce that the there are no linear relations on the f-vectors of simple d-polytopes other than the Dehn-Sommerville equations.

(10 points)

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Exercise 5. Let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_d)$ be a permutation of $[d]$. An index $1 \le i \le d$ is a *descent*, if $\sigma_i < \sigma_{i-1}$.

Let $\Pi_{d-1} = \Pi(1, 2, \dots, d)$ be the $(d-1)$-dim permutahedron.

i) Show that $h_k(\Pi_{d-1})$ is the number $E(n, k)$ of permutations with exactly k descents.
[Hint: Pick a good linear function $\ell(x)$.]

ii) Deduce that $E(d, k) = E(d, d-k+1)$.

iii) (Bonus) Can you determine $f(\Pi_{d-1})$ from that knowledge?

(10+3 points)