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Discrete Geometry I

Homework # 5 — due November 28

Two weeks time for this sheet. Please mark **three** problems that will be graded.

Exercise 1. Let $Q = P_1 * P_2$ be the join of two polytopes. Show that

i) faces of Q are in bijection with joins $F_1 * F_2$ of faces of P_1 and P_2;

ii) $f^0_Q(t) = f^0_{P_1}(t) \cdot f^0_{P_2}(t)$, where $f^0(P) := \sum_{i=-1}^{d} f_i t^{i+1}$;

iii) $Q^\triangle$ is combinatorially isomorphic to $P_1^\triangle * P_2^\triangle$.

(10 points)

Exercise 2. i) Let $T(x) = Ax$ be a linear transformation and $P \subset \mathbb{R}^d$ a d-polytope with $0 \in \operatorname{int}(P)$. Show that

$$T(P)^\triangle \;=\; T^*(P^\triangle)$$

where $T^*(x) = (A^t)^{-1}x$.

ii) Let $P, Q \subset \mathbb{R}^d$ polytopes with $\operatorname{relint}(P) \cap \operatorname{relint}(Q) = \{0\}$ and let $P \oplus Q = \operatorname{conv}(P \cup Q)$ be their direct sum. Show that $(P \oplus Q)^\triangle$ is affinely isomorphic to $P^\triangle \times Q^\triangle$.

iii) Let $P \times Q$ be a product of polytopes and $(u, v) \in V(P \times Q)$ a vertex. What is the vertex figure $(P \times Q)/(u, v)$ in terms of the vertex figures P/u and Q/v?

(10 points)

Exercise 3. i) Show that the direct sum is a projection of the join.

ii) Show that the product is a section of the join.

iii) Show that the cyclic polytope $C_4(6)$ is the direct sum of two triangles.

(10 points)

Exercise 4. Can $(12, 31, 30, 11)$ be the f-vector of

i) a simple/simplicial polytope?

ii) a prism?

iii) a pyramid?

iv) a join?

v) (Bonus) any polytope?

(10 points)

(continued on backside)

Exercise 5. For a polyhedron $Q \subset \mathbb{R}^d$, the definition of vertex-edge-graph $G(Q)$ of Q makes sense. But $G(Q)$ might have no vertices (e.g., if Q is not pointed) or might not have edges (e.g., if Q is a pointed cone). Let Q be a pointed unbounded polyhedron with at least two vertices.

i) Show that $G(Q)$ is connected.

ii) Give an example of a 3-dimensional polyhedron whose graph is not 2-connected.

iii) What about higher dimensions?

(10 points)

Exercise 6. Define the *Cartesian product* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ as $\Gamma_1 \times \Gamma_2 = (V_1 \times V_2, E)$, where two vertices (v_1, v_2) and (w_1, w_2) are adjacent $\Leftrightarrow$ ($v_1 = w_1$ and v_2 is adjacent to w_2 in Γ_2) or ($v_2 = w_2$ and v_1 is adjacent to w_1 in Γ_1).

i) Let Γ_1 and Γ_2 be the graphs of polytopes P_1 and P_2, respectively. Show that $\Gamma_1 \times \Gamma_2$ is the graph of $P_1 \times P_2$.

ii) Give an example of a Cartesian product $\Gamma = \Gamma_1 \times \Gamma_2$, where Γ and Γ_1 are graphs of polytopes, but Γ_2 is not.

iii) (Bonus) Can you find two graphs Γ_1, Γ_2 that are both not graphs of polytopes but $\Gamma_1 \times \Gamma_2$ is?

(10+3 points)

Exercise 7. Let $P = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a bounded, possibly empty polyhedron such that $A \in \mathbb{R}^{m \times d}$ has full rank d. Consider the polyhedron

$$\widehat{P} \;=\; \{(x,t) \in \mathbb{R}^{d+1} : Ax + t\,b \leq b, 0 \leq t \leq 1\}$$

i) Show that $v = (0, 0, \ldots, 0, 1)$ is a vertex of $\widehat{P}$.

ii) Show that a vertex $(u, t) \in V(\widehat{P})$ corresponds to a vertex of P iff $t = 0$.

iii) Let $(p_0, t_0) \in \widehat{P}$ be a point with t_0 minimal. Deduce that $P = \emptyset$ iff $t_0 > 0$.

(10 points)