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Discrete Geometry I

Homework # 4 — due November 14th

Exercise 1. Let P be a polytope with vertices $v_1, \dots, v_n$ and facets $F_1, \dots, F_m$. The *vertex-facet incidence matrix* is given by $M = M(P) \in \{0,1\}^{m \times n}$ with $M_{ij} = 1$ iff $v_j \in F_i$.

i) What does $M(P^\triangle)$ look like?

ii) How can you tell from M if P is simple or simplicial?

iii) How can you reconstruct the face lattice $\mathcal{L}(P)$ from M? Can you determine the dimension of P from M?

(10 points)

Exercise 2. Two polytopes P_1, P_2 are called *combinatorially isomorphic* if their face lattices are isomorphic.

i) Find two simplicial 3-polytopes with 6 vertices each that are not combinatorially isomorphic.

ii) Find two combinatorially non-isomorphic 3-polytopes P_1 and P_2 with the same sets of vertex figures. More exactly, there must exist a bijection $\iota \colon V_1 \to V_2$ between the vertex sets of P_1 and P_2 such that the vertex figure of v is combinatorially isomorphic to the vertex figure of $\iota(v)$, for all $v \in V_1$.

iii) Find two combinatorially non-isomorphic 4-polytopes P_1 and P_2 with the same sets of vertex figures.

(10 points)

Exercise 3. A full-dimensional polytope $P \subset \mathbb{R}^d$ is called *inscribed* in the unit sphere if $\|v\| = 1$ for all vertices $v \in V(P)$ and *circumscribed* if every facet $F \subset P$ contains a unique $p \in \operatorname{relint} F$ with $\|p\| = 1$. Assume that $0 \in \operatorname{int} P$.

i) Show that if P is inscribed in the unit sphere, then $P^\triangle$ is circumscribed.

ii) A face F of P is called tangent to the unit sphere if $\min\limits_{x \in F} \|x\| = 1$. Show: if all k-dimensional faces of P are tangent to the unit sphere, then all $(d-k-1)$-dimensional faces of $P^\triangle$ are also tangent to the unit sphere.

iii) Assume tht all k-faces are tangent to the unit sphere. Show that $P \cap P^\triangle$ is inscribed in the unit sphere and $\operatorname{conv}(P \cup P^\triangle)$ is circumscribed about it.

(10 points)

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Exercise 4. Let P be a d-dimensional polytope.

i) Show that among the vertices of P one can choose $d+1$ whose join in the face lattice (i. e. the smallest face containing all of them) is P.

ii) Show that among the facets of P one can choose $d+1$ whose intersection is empty.

iii) Let $-1 \leq k < h \leq d-1$. Show that for every k-face F of P one can find $h-k+1$ faces of dimension h whose intersection is F. (Hint: Start with the case $h = d-1$.)

(10 points)