Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Ivan Izmestiev

WS2012/13

Discrete Geometry I

Homework # 3 — due November 7th

Please work in *pairs*. Try to solve all the problems but mark *two* solutions. Only these will be graded. You can earn *20 points* on every homework sheet (10 per exercise). You can get extra credit by solving the bonus problems. *State* who wrote up the solution.

Exercise 1. Let $C = \{x \in \mathbb{R}^d : a_i^t x \leq 0 \quad i = 1, \ldots, m\}$ be a convex polyhedral cone.

i) Show that if $C \cup (-C) = \mathbb{R}^d$, then $C = \mathbb{R}^d$ or C is a halfspace.

ii) Infer that, if C is bounded by $m \geq 2$ irredundant halfspaces, then any $r \notin C \cup (-C)$ has the property that $a_i^t r \neq 0$ for all i and $a_j^t r > 0$ and $a_k^t r < 0$ for some j, k.

(10 points)

Exercise 2. Let $C = \mathrm{cone}(U) \subseteq \mathbb{R}^d$ be a convex polyhedral cone (U finite).

i) Prove that C is pointed if and only if 0 is a face, that is, $C \cap H = \{0\}$ for some supporting hyperplane $H = \{x : c^t x = 0\}$.

ii) With this hyperplane show that $C_p = C \cap (p + H)$ is a polytope for any $p \in C \setminus \{0\}$.

iii) Deduce that there is a unique inclusion-minimal subset $U' \subseteq U$ such that $C = \mathrm{cone}(U)$.

(10 points)

Exercise 3. i) Let $Q = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a polyhedron. Show that $\mathrm{lineal}(Q) = \ker A$.

ii) Let $K \subset \mathbb{R}^d$ be a closed convex set and $L := \mathrm{lineal}(K)$. Show that for any point $p \in K$ the convex set $\overline{K_p} := K \cap (p + L^\perp)$ is pointed and that $K = \overline{K_p} + L$.

(10 points)

Exercise 4. Let $A, B \subseteq \mathbb{R}^d$ be convex polyhedral cones. Show that

i) $A \subseteq B \Rightarrow A^\circ \supseteq B^\circ$

ii) $(A \cup B)^\circ \;=\; A^\circ \cap B^\circ$

iii) If A is a linear subspace of $\mathbb{R}^d$, then $A^\circ = A^\perp$.

iv) A is pointed if and only if A° is full-dimensional (i.e. $\dim A = d$).

(10 points)

(continued on backside)

Exercise 5. Let $C \subset \mathbb{R}^d$ be a full dimensional convex polyhedral cone and let $\mathbb{R}^m_{\geq 0} = \{x \in \mathbb{R}^m : x_i \geq 0 \text{ for all } i = 1, \dots, m\}$.

i) Show that there is an n and a linear map $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that $C = \pi(\mathbb{R}^n_{\geq 0})$.

ii) Let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be a surjective linear map (a projection) and let $L = \ker \pi$. Show that

$$\pi(C)^\circ \;\cong\; C^\circ \cap L^\perp.$$

iii) Show that there is some $m \geq 0$ and a linear subspace $L \subseteq \mathbb{R}^m$ such that C is linearly isomorphic to $\mathbb{R}^m_{\geq 0} \cap L$.

(10 points)