Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Ivan Izmestiev

WS2012/13

Discrete Geometry I

Homework # 2 — due October 31th

Please work in *pairs*. Try to solve all the problems but mark *two* solutions. Only these will be graded. You can earn *20 points* on every homework sheet (10 per exercise). You can get extra credit by solving the bonus problems. *State* who wrote up the solution.

Please write your full name, student id (Matrikel nummer), semester and 'category' (e.g. Math, computer science; MSc, BSc, Diplom).

Exercise 1. i) Let P be a k-neighborly polytope with the vertex set $\mathcal{V}$. Show that for every $\mathcal{V}' \subset \mathcal{V}$ the convex hull $\operatorname{conv}(\mathcal{V}')$ is also k-neighborly.

ii) Show using the Radon Lemma: if in a 3-dimensional polytope each pair of vertices is joined by an edge, then this polytope is a simplex.

iii) Prove that in a k-neighborly polytope every $(2k-1)$-face is a simplex.

(10 points)

Exercise 2. For $d < n$ and $r_1 < r_2 < \cdots < r_n$ let

$$C = \mathsf{Cyc}_d(r_1, r_2, \dots, r_n) = \operatorname{conv}\{\gamma_d(r_i) : i = 1, 2, \dots, n\}$$

be a cyclic polytope. Here $\gamma_d(t) = (t, t^2, \dots, t^d)^t \in \mathbb{R}^d$.

i) For a subset $I \subseteq [n]$ an element $j \notin I$ is an *odd* or *even gap* if the number $|\{i \in I : i < j\}|$ is odd or even, respectively.
Show that for $I \subseteq [n]$ with $|I| = d$ we have that

$$F_I = \operatorname{conv}\{\gamma_d(r_i) : i \in I\}$$

is a facet of C if and only if all gaps of I are either even or odd.

ii) Assume that d is even. Let $F \subset C$ be a facet and $H = \{a^t x = b\}$ its (unique) supporting hyperplane. F is called a *lower facet* with respect to the coordinate direction x_d if $a_d < 0$. Show that $F = F_I$ is lower if and only if I has only even gaps.

iii) (Bonus) What is the number (closed formula!) of facets $f_{d-1}(C)$? What is the number of k-faces?

(10+3 points)

(continued on backside)

Exercise 3. For $n \geq 3$ let

$$P_n \;=\; \operatorname{conv}\left\{(\cos\tfrac{2\pi k}{n}, \sin\tfrac{2\pi k}{n}) : k = 1, 2, \ldots, n\right\} \subset \mathbb{R}^2$$

be the *regular n-gon*.

i) Show that if n is even, then P_n is a *zonotope*, that is,

$$P_n = [w_1, z_1] + [w_2, z_2] + \cdots + [w_m, z_m]$$

for some $w_1, z_1, \ldots, w_m, z_m \in \mathbb{R}^2$.

ii) (Bonus) Can the regular 9-gon be represented as the Minkowski sum of a triangle and a hexagon? Can the regular 7-gon be represented as the Minkowski sum of two polygons that are not similar to each other?

(10+3 points)

Exercise 4. Recall that for $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$, the *generalized permutahedron* is defined by

$$\Pi_{d-1}(a) \;=\; \operatorname{conv}\{(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(d)}) : \pi \text{ permutation of } [d]\ \}$$

i) Give an example of a such that $a_i \neq a_j$ for all $i \neq j$ and $\Pi_{d-1}(a)$ is not a zonotope. (*Hint:* $d = 3$ suffices.)

ii) Prove that if $a_1 \geq a_2 \geq \cdots \geq a_d$ and $b_1 \geq b_2 \geq \cdots \geq b_d$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_d b_d \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_d b_{\pi(d)}$$

for every permutation π of $[d]$.

From here on assume that $a = (a_1 > a_2 > \cdots > a_d)$.

iii) For a permutation π, denote by $\pi(a) = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(d)})$. Show that $[a, \pi(a)] = \operatorname{conv}\{a, \pi(a)\}$ is an edge of $\Pi_{d-1}(a)$ if π is an *adjacent transposition*, that is, $\pi(a) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_d)$ for some $1 \leq i < d$. Deduce that $\Pi_{d-1}(a)$ is simple.

(10 points)