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Discrete Geometry I

Homework # 1 — due October 24th

Please work in *pairs*. Try to solve all the problems but mark *two* solutions. Only these will be graded. You can earn *20 points* on every homework sheet (10 per exercise). You can get extra credit by solving the bonus problems. *State* who wrote up the solution.

Please write your full name, student id (Matrikel nummer), semester. and 'category' (e.g. Math, computer science; MSc, BSc, Diplom).

Exercise 1. Let P_1 and P_2 be polytopes.

i) Show that for every pair of faces $F_1 \subseteq P_1, F_2 \subseteq P_2$ their product $F_1 \times F_2$ is a face of $P_1 \times P_2$. And conversely, for every non-empty face F of P there are unique faces $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$ such that $F = F_1 \times F_2$.

ii) Let $\pi : P \to Q$ be a projection of polytopes. Show that $\pi^{-1}(F)$ is a face of P for every face $F \subseteq Q$.

iii) Show that if $K_1, K_2 \subseteq \mathbb{R}^d$ are convex, then so is $K_1 + K_2$. Stronger even, show that $P + Q$ is a polytope whenever P and Q are.
[Bonus: Show that the reverse is also true: if the Minkowski sum of two convex sets is a polytope, then both summands are polytopes.]

iv) Show that if $F \subseteq P_1 + P_2$ is a face, then $F = F_1 + F_2$ for some faces $F_i \subseteq P_i$. Show that F_1 and F_2 are unique.

(10+3 points)

Exercise 2. Let $C_d = [-1, +1]^d$ be the d-cube.

i) For every vector $c \in \mathbb{R}^d$, describe the face maximizing $\ell(x) = c^t x$. In particular, which face of the eight-dimensional cube maximizes the scalar product with the vector $c = (1, -7, 1, 0, 2, 0, 1, -2)^t$?

ii) Establish a bijection r between non-empty faces of C_d and the set $\{-, 0, +\}^d$. Show that

$$\dim F \;=\; \#\{i : r(F)_i = 0\}$$

and give a formula for the number f_i of i-dimensional faces of C_d.

(10 points)

(continued on backside)

Exercise 3. i) The d-dimensional *crosspolytope* is

$$C_d^{\triangle} = \operatorname{conv}\{\pm e_1, \pm e_2, \ldots, \pm e_d\}$$

For any $u, v \in \{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ show that $[u, v] = \operatorname{conv}\{u, v\}$ is an edge of $C_d^{\triangle}$ if $u \neq \pm v$.

ii) A polytope P is *centrally-symmetric* if $-P = P$. Show that $P = \operatorname{conv}(V)$ is centrally symmetric if and only if P is a projection of the n-dim'l crosspolytope $C_n^{\triangle}$ with $n = \frac{1}{2}|V|$.

iii) (Bonus) Let $P = \operatorname{conv}\{v_1, v_2, \ldots, v_n\}$ be a d-polytope.
For a point $p \in \operatorname{int} P$ show that $p \in \operatorname{int} \operatorname{conv}(V')$ for a subset $V' \subseteq V(P)$ of cardinality $\leq 2 \cdot \dim P$. Show that there are polytopes and points for which this bound is sharp.

(10+3 points)

The computer program `polymake` allows to analyze combinatorial properties of polytopes. It is called on UNIX-computers at FU by the command line

`/import/polymake/bin/polymake`

An introduction tutorial for polymake can be found at

`http://polymake.org/doku.php/tutorial/intro_tutorial`

Exercise 4. The *permutahedron* $\Pi_{d-1} \subset \mathbb{R}^d$ is defined as the convex hull of all vectors obtained by permuting the coordinates of the vector $(1, 2, \ldots, d)^t$. In `polymake`, Π_3 is denoted by `permutahedron(4)`.

Let $a_1 \geq a_2 \geq a_3 \geq a_4$ be real numbers. The *generalized permutahedron* (or *orbit polytope*) $\Pi_3(a_1, a_2, a_3, a_4)$ is the convex hull of all the vectors in $\mathbb{R}^4$ given by permutations of the coordinates $(a_1, a_2, a_3, a_4)^t$.

i) Using polymake, compute the f-vector of Π_3. Does this tell you the dimension of Π_3? Why does Π_3 have this dimension?

ii) Study the f-vectors of generalized permutahedra $\Pi_3(a_1, a_2, a_3, a_4)$ experimentally. What are the 6 possible f-vectors? What are the possible numbers of vertices of $\Pi_3(a_1, a_2, a_3, a_4)$?
Data of a polytope can be entered by the command

`$p=load('yourfile');`

where `yourfile` can be a plain text file looking as

POINTS
1 a_1 a_2 a_3 a_4
1 a_1 a_2 a_4 a_3
...

iii) What are possible values of $\dim \Pi_3(a_1, a_2, a_3, a_4)$, and what are the corresponding conditions on (a_1, a_2, a_3, a_4)?

iv) Show that $\Pi_3(-1, -1, 1, 1)$ is linearly isomorphic to the octahedron (or 3-dimensional crosspolytope; See Exercise 3).

(10 points)