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Discrete Geometry I

Homework # 13 — due February 4th

Please mark **one** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be an arrangement of n distinct linear hyperplanes in $\mathbb{R}^d$ and define the following arrangements of $n-1$ hyperplanes in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$, respectively:

$$\mathcal{H} \setminus H_1 = \{H_2, \ldots, H_n\}$$
$$\mathcal{H}|_{H_1} = \{H_1 \cap H_2, \ldots, H_1 \cap H_n\}.$$

i) Let $\mathsf{r}(\mathcal{H})$ be the number of full-dimensional regions of $\mathcal{H}$. Prove that

$$\mathsf{r}(\mathcal{H}) = \mathsf{r}(\mathcal{H} \setminus H_1) + \mathsf{r}(\mathcal{H}|_{H_1}).$$

ii) What is the maximal number of vertices that a zonotope on n segments in $\mathbb{R}^d$ can have?

(10 points)

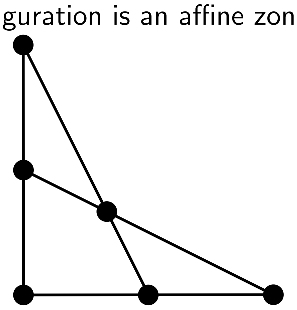
Exercise 2. Let $\bar{Z} = (\bar{z}_1, \ldots, \bar{z}_n) \in \mathbb{R}^{(n-d)\times n}$ be the **zonal diagram** of a d-zonotope $\mathcal{Z} = \sum[-z_i, z_i]$.

i) Explain the effect of replacing $\bar{z}_i$ by $-\bar{z}_i$ in a zonal diagram.

ii) Let $c \in \mathbb{R}^{n-d}$ be a generic vector (such that $c^t\bar{z}_i \neq 0$ for all $i \in [n]$). By (i), we can assume without loss of generality that $c^t\bar{z}_i > 0$ for all $i \in [n]$. Let $\tilde{z}_i = \frac{\bar{z}_i}{c^t\bar{z}_i}$. We can identify $\tilde{Z} = (\tilde{z}_1, \ldots, \tilde{z}_n)$ with an ordered collection of n vectors in $\mathbb{R}^{n-d-1}$. ($\tilde{Z}$ is called the **affine zonal diagram** of $\mathcal{Z}$.)
For $J = (J^+, J^-)$ (with $J^+, J^- \subseteq [n]$, $J^+ \cap J^- = \emptyset$) give a necessary and sufficient condition for $F(J)$ being a face of $\mathcal{Z}$ in terms of $\tilde{Z}$.

iii) When is $\mathcal{Z}$ a simple polytope (in terms of $\tilde{Z}$)?

iv) The following point configuration is an affine zonal diagram.



Compute the f-vector of $\mathcal{Z}$.

v) [Bonus:] Draw the graph of $\mathcal{Z}$.

(10+3 points)

Bonus Exercise 1. Find a shelling for the boundary complex of the d-crosspolytope and compute its h-vector.

(+5 points)

Bonus Exercise 2. Assume for the moment that

$$f = f(P) = (1, 15, 34, 28, 9, 1)$$

is the f-vector of a polytope P.

i) What is the dimension of P?

ii) Is P simple or simplicial?

iii) Could P be a prism?

iv) Could P be a product of polytopes?

v) Could P be a direct sum of polytopes (that contain the origin)?

vi) Could P be a pyramid?

vii) Could P be a join?

viii) Could P be a stacked polytope?

ix) Could P be a truncation polytope?

x) Could P be a neighborly polytope?

(+10 points)