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Discrete Geometry I

Homework # 12 — due January 28th

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Let $G = (g_1, \dots, g_n) \in \mathbb{R}^{(n-d-1)\times n}$ be the Gale transform of a d-polytope with n vertices.

i) Show that $G' = \left(\frac{1}{2}g_1, \frac{1}{2}g_1, g_2, \dots, g_n\right)$ is the Gale transform of a polytope and determine its combinatorics.

ii) Let $G_1 = (g_1, \dots, g_{n_1}) \in \mathbb{R}^{(n_1-d_1-1)\times n_1}$ and $G_2 = (h_1, \dots, h_{n_2}) \in \mathbb{R}^{(n_2-d_2-1)\times n_2}$ be the Gale transforms of a d_1-polytope P_1 and a d_2-polytope P_2, respectively. Describe the polytope for which

$$G = \begin{pmatrix} g_1 & \cdots & g_{n_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & h_1 & \cdots & h_{n_2} \end{pmatrix}$$

is a Gale transform.

(10 points)

Exercise 2. i) Let P and Q be combinatorially equivalent d-polytopes with $d+2$ vertices. Prove that there is a projective transformation T, admissible for P, such that $T(P) = Q$. (This means that d-polytopes with $d+2$ vertices are projectively unique.)

ii) Use Gale transforms to construct two combinatorially equivalent d-polytopes with $d+3$ vertices that are not projectively equivalent.

iii) Let U and V be two configurations of $d+2$ points that affinely span $\mathbb{R}^d$ and such that there is no hyperplane that contains all the points of U (resp. V) but one. Prove that there is a projective transformation T such that $T(U) = V$.

(10 points)

Exercise 3. Let P be a d-polytope with vertices $(v_1, \dots, v_n) \in \mathbb{R}^{d\times n}$ that is not a pyramid, and let $\bar{G} = (\bar{g}_1, \dots, \bar{g}_n) \in \mathbb{R}^{(n-d-2)\times n}$ with the partition (B, W) be its affine Gale transform.

i) Prove that, for $I \subseteq [n]$, the convex hull of $\{v_i \ : \ i \in I\}$ is a face of P if and only if

$$\text{relint conv}(\bar{g}_i : i \in B \setminus I) \cap \text{relint conv}(\bar{g}_i : i \in W \setminus I) \neq \emptyset.$$

ii) Prove that, for disjoint subsets $I, J \subseteq [n]$

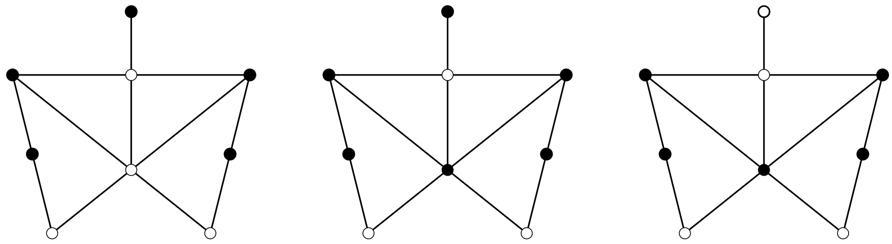
$$\text{relint conv}(v_i \ : \ i \in I) \cap \text{relint conv}(v_i \ : \ i \in J) \neq \emptyset$$

if and only if there is an affine hyperplane $H \subset \mathbb{R}^{n-d-2}$ such that

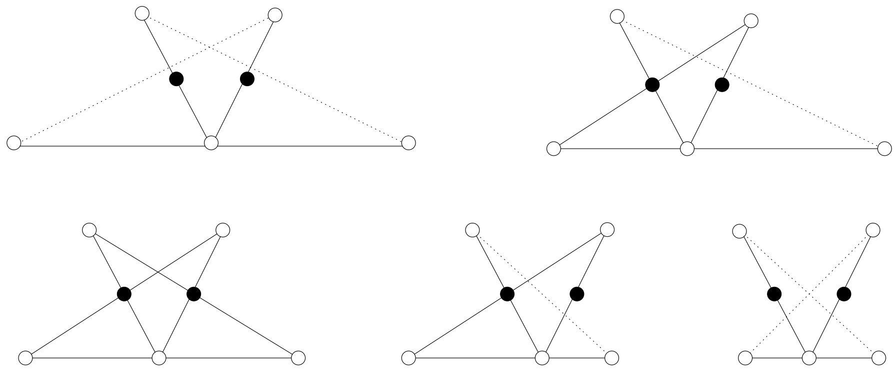
$\bar{G} \setminus H^- = \{\bar{g}_i \ : \ i \in I \cap B\} \cup \{\bar{g}_i \ : \ i \in J \cap W\}$, and
$\bar{G} \setminus H^+ = \{\bar{g}_i \ : \ i \in J \cap B\} \cup \{\bar{g}_i \ : \ i \in I \cap W\}$,

(10 points)

Exercise 4. i) Which of the following point configurations are affine Gale diagrams of (sets of vertices of) polytopes? Why, or why not?



ii) Consider the following diagrams. Verify that they are affine Gale diagrams of 3-polytopes and construct the associated polytopes (draw their graph).



(10 points)

Bonus Exercise. Prove that for any set U of at most $2d+1$ distinct points in general position in $\mathbb{R}^d$ there is a projective transformation T, admissible for U, such that $T(U)$ is in convex position.
[Hand in the exercise even if you can only prove it for some values of d.]

(+ 10 points)