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Discrete Geometry I

Homework # 11 — due January 21st

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Which of the following are h-vectors of simple polytopes?

i) $h^1 = (1, 2, 2, 1)$

ii) $h^2 = (1, 2, 4, 1)$

iii) $h^3 = (1, 0, 3, 0, 1)$

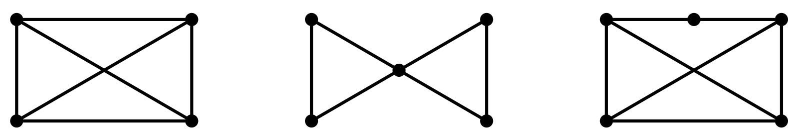
iv) $h^4 = (2, 3, 5, 3, 2)$

If not, why? If yes, give an example of a polytope.

(10 points)

Exercise 2. i) Write a function that computes the rigidity matrix of a framework.

ii) Decide for each of the following three planar frameworks, whether it is rigid and whether it is infinitesimally rigid.



iii) [Bonus] True or false: Every simple 3-polytope different from the simplex has a rigid realization.

(10+3 points)

Exercise 3. i) Let $P = \text{conv}(v_1, \ldots, v_n) \subset \mathbb{R}^d$ be a full-dimensional polytope. Prove that $P' = \text{conv}(v_1, \ldots, v_n, v_{n+1}) \subset \mathbb{R}^{d+1}$ is the pyramid over P if and only if any Gale transform of P' is of the form $G' = (g_1, \ldots, g_n, 0)$ and $G = (g_1, \ldots, g_n)$ is a Gale transform of P.

ii) For $m, n \geq 2$ consider

$$G_{m,n} = (\underbrace{\tfrac{1}{n}, \ldots, \tfrac{1}{n}}_{n \text{ times}}, \underbrace{\tfrac{-1}{m}, \ldots, \tfrac{-1}{m}}_{m \text{ times}}).$$

Prove that G is the Gale transform of a polytope $P_{m,n}$. Describe $P_{m,n}$. [Hint: Think about direct sums.]

iii) How many combinatorially distinct d-polytopes with $d+2$ vertices are there? [Hint: In dimension $d = 3$ there are two.]

(10 points)

Exercise 4. Each of the following vector configurations V_i is the Gale transform of a polytope P_i:

$$V_1 = \begin{pmatrix} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 1 & -1 & 1 & -2 & 1 & -1 & 1 \end{pmatrix};$$

$$V_2 = \begin{pmatrix} -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & -2 & -1 & 1 & 1 \end{pmatrix};$$

$$V_3 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 2 & 1 & -2 & -2 & -2 & 1 & 2 \end{pmatrix};$$

$$V_4 = \begin{pmatrix} -2 & -1 & 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & -3 & 2 & -3 & 1 & 1 \end{pmatrix}$$

i) (🖱?) Compute the number of facets of P_i.

ii) For which i is P_i simplicial?

iii) For which i is P_i neighborly?

iv) For which pairs (i, j) is $P_i \cong P_j$?

(10 points)