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Discrete Geometry I

Homework # 10 — due January 14th

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Let $\Delta_{m,n} \subset 2^{[m]\times[n]}$ be a $m \times n$ - chessboard complex for $m \leq n$.

i) What is the dimension of $\Delta_{m,n}$?

ii) What is its f-vector?

iii) For which (m,n) is $\Delta_{m,n}$ a pseudomanifold?

iv) Prove $\mathrm{lk}_{\Delta_{m,n}}(v) \cong \Delta_{m-1,n-1}$ for every vertex $v \in [m] \times [n]$.

v) Is $\Delta_{3,4}$ the boundary complex of a polytope?

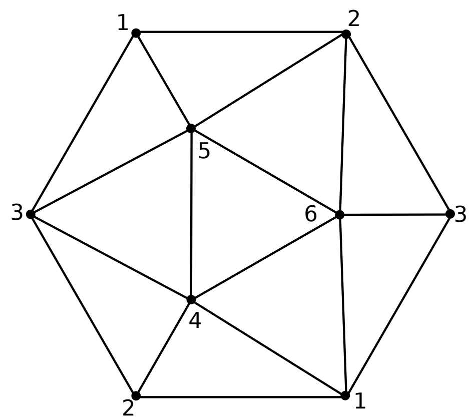
vi) [Bonus: What is $\Delta_{3,4}$?]

(10+3 points)

Exercise 2. Let $P \subset \mathbb{R}^d$ be a simple polytope with 0 in the interior. Pick $c \in \mathbb{R}^d$ general with respect to P. Let $v_1, \ldots, v_m$ be a labelling of the vertices of P such that $i < j$ implies $c^t v_i < c^t v_j$. Show that this gives a shelling order of the facets $F_i = v_i^\diamond$ of $P^\triangle$.

(10 points)

Exercise 3. Consider the following 2-dimensional simplicial complex Δ on 6(!) vertices. (Mind the identifications on the boundary!).



i) Compute $h(\Delta)$.

ii) Is Δ partitionable?

iii) Is Δ shellable?
[Hint: look at the last simplex in a potential shelling.]

(10 points)

Exercise 4. Let $P \subset \mathbb{R}^d$ be a d-polytope with facets $F_1, \ldots, F_m$ and correspoding supporting hyperplanes $H_i = \{x : a_i^t x = b_i\}$. A point $q \in \mathbb{R}^d$ is *beneath* F_i if $a_i^t q < b_i$ and *beyond* F_i if $a_i^t q > b_i$. Let $k \in [m]$ be fixed.

i) Show that there is a point $q_k \in \mathbb{R}^d$ such that q_k is beyond F_k and beneath F_i for all $i \neq k$. (Hint: Start from a well chosen point in F_k)

The operation of *stacking a vertex* onto the facet F_k of P is the polytope

$$\mathrm{stack}(P, F_k) \;=\; \mathsf{conv}(P \cup \{q_k\})$$

where q_k is as defined above.

ii) Show that 'stacking a facet' is dual to 'truncating a vertex', i.e.

$$\mathrm{trunc}(P, v)^{\triangle} \;\cong\; \mathrm{stack}(P^{\triangle}, v^{\diamond})$$

In particular, the combinatorial type of $\mathrm{stack}(P, F_k)$ is independent of q_k. [Hint: Put P into the 'right position' and use polarity.]

iii) A d-dimensional *stacked polytope* on n vertices is the $(n-d-1)$-fold stacking of a d-simplex. Show that the f-vector is independent of the stacking order. Give an example of two stacked 3-polytopes on 7 vertices that are combinatorially distinct.

(10 points)

Bonus Exercise. Show that every d-dimensional simplicial complex can be embedded in $\mathbb{R}^{2d+2}$. [Hint: embed it in the boundary of a suitable simplicial $(2d+2)$-polytope.]

(+ 3 points)