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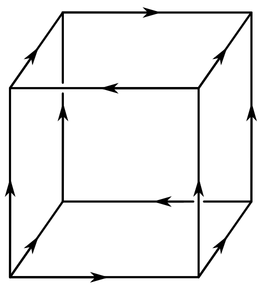
Discrete Geometry I

Homework # 9 — due January 7th

Please mark **three** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Consider the following acyclic orientation of the graph of the 3-cube $C_3 = [0,1]^3$.



i) Show that the orientation $\mathcal{O}$ is *good*.

ii) Show that there is no $c \in \mathbb{R}^3$ such that $\mathcal{O} = \mathcal{O}_c$.
[Bonus: Is there a 3-polytope combinatorially isomorphic to C_3 such that $\mathcal{O} = \mathcal{O}_c$?]

iii) An orientation $\mathcal{O}$ is called a **unique sink orientation** (USO) of $G(P)$, if $G(F)$ has a unique sink for every face $F \subset P$. Is every USO also acyclic?

(10+3 points)

Exercise 2. Consider the graph $G = G(P) = (V, E)$ of a simple 4-polytope P with vertices $V = \{0, \dots, 8\}$ and edges

$$\begin{aligned} E \;=\; \{&(0,1),(0,2),(0,4),(0,7),(1,2),(1,5),(1,8),(2,3),(2,6),\\ &(3,4),(3,5),(3,6),(4,5),(4,7),(5,8),(6,7),(6,8),(7,8)\}. \end{aligned}$$

i) What is $s(P)$, the total number of nonempty faces of P?

ii) Let G_1 and G_2 be the subgraphs of P induced by the vertex sets $V_1 = \{0,1,4,5,7,8\}$ and $V_2 = \{0,1,2,3,5,7,8\}$, respectively. Are G_1 and G_2 subgraphs corresponding to facets of P?

You can find these graphs at:
`http://page.mi.fu-berlin.de/sanyal/teaching/dg1/Hw9Ex2.sws`

(10 points)

Exercise 3. Let $C \subset \mathbb{R}^d$ be a pointed cone.

i) Show that there is a hyperplane H such that for every k-face $F \subset C$ we have that $H \cap F$ is a polytope of dimension $k-1$.

ii) Show that

$$\chi(C) \;:=\; \sum_{i \geq 0} (-1)^i f_i(C) \;=\; 0.$$

iii) [Bonus: What is $\chi(Q)$ for a nonempty polyhedron Q?]

(10+3 points)

Exercise 4. Let P be a simple $(d-1)$-polytope. Let $P' = \mathrm{prism}(P) := P \times [0,1]$ be the prism over P.

i) Show that $h_i(P') = h_i(P) + h_{i-1}(P)$ for $i = 0, \dots, d+1$ only using orientations on graphs. (Here, $h_{-1}(P) = h_d(P) = 0$.)

Let $d = 2m$ be even and for $0 \le i \le m$ define $Q_i^d := \mathrm{prism}^{d-2i}(\Delta_{2i})$, the $(d-2i)$-fold prism over the $2i$-dimensional simplex.

ii) Compute the h-vector of Q_i^d.

iii) Show that the h-vectors $(h(Q_i^d) : 0 \le i \le \lfloor \frac{d}{2} \rfloor)$ are affinely independent.

(10 points)

Exercise 5. i) Show that if P is a simple and simplicial d-polytope for $d \ge 3$, then P is a simplex.

ii) Show that there is no 4-polytope with $f_0 = 6$ and $f_1 = 12$.
[Hint: Try to infer the f-vector from this data.]

iii) Give an example of a 4-connected 4-regular graph which is not the graph of a simple 4-polytope. [Hint: 6 vertices suffice.]

(10 points)

Exercise 6. Let P be a simple 4-dimensional polytope and consider

$$K(P) \;:=\; \sum_{F \subset P \text{ facet}} (f_0(F) - f_1(F) + f_2(F))$$

i) Show that $K(P) = 2f_3(P)$.

ii) Expand the sum and interpret each of the 4 sums individually to show that $K(P) = 4f_0(P) - 3f_1(P) + 2f_2(P)$.

iii) Together this gives a linear relation on the f-vector. How is it implied by the Dehn-Sommerville equations?

(10 points)

Bonus Exercise 1. Let $P \subset \mathbb{R}^d$ be a d-polytope with vertices $v_1, \dots, v_n$. Let $G = ([n], E)$ be the graph with $ij \in E$ if $[v_i, v_j]$ is a face of P.

i) Define the zonotope $Z_P \subset \mathbb{R}^d$ as

$$Z_P \;:=\; \sum_{ij \in E} [v_i - v_j, v_j - v_i].$$

Prove that the vertices of Z_P are in bijection with the orientations $\mathcal{O}_c$ for $c \in \mathbb{R}^d$ general with respect to P.

ii) Define the zonotope $Z_G \subset \mathbb{R}^n$ by

$$Z_G \;:=\; \sum_{ij \in E} [e_i - e_j, e_j - e_i].$$

Prove that the vertices of Z_G are in bijection with acyclic orientations of G.

iii) Prove that Z_P is a projection of Z_G.

iv) Prove that the permutahedron Π_{d-1} is affinely equivalent to Z_G for some graph G.

v) Compute the h-vector of Π_{d-1}.

(+3+3+1+3+3 points)

Bonus Exercise 2. Let P be a centrally-symmetric d-polytope, that is, $P = -P$. Show that

$$s(P) \ := \ f_0 + f_1 + \cdots + f_d \ \geq \ 3^d$$

i) for $d = 2$ (trivial);

ii) for $d = 3$ (interesting);

iii) for $d = 4$ (challenging!);

iv) for $d \geq 5$ (open!!).

(+3+9+27+81 points)