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WS2014/15

Discrete Geometry I

Homework # 8 — due December 10th

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Consider the set

$$\mathcal{F}_3 = \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2, f_0 \leq 2f_2 - 4, f_2 \leq 2f_0 - 4\}$$

i) Show that for every 3-polytope P, its f-vector $f(P)$ belongs to $\mathcal{F}_3$.

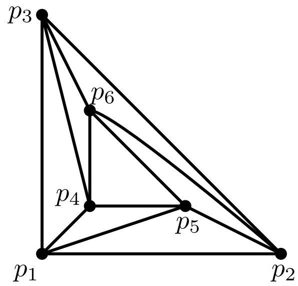
ii) (🖳?) Draw $\mathcal{F}_3$ (for $f_0, f_2 \leq 10$), identify the points corresponding to the f-vectors of the simplex, the cube and the octahedron, and those corresponding to pyramids over polygons.

iii) Describe the effect of truncating (see exercise 4 of homework sheet # 7) and stacking (its polar operation) on the f-vector of a 3-polytope.

iv) Show that every $f \in \mathcal{F}_3$ is the f-vector of a 3-polytope.

(10 points)

Exercise 2. Consider the following planar graph G



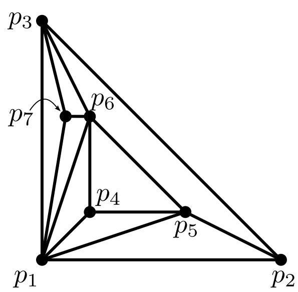
embedded in the plane with $p_1 = (0, 0)$, $p_2 = (5, 0)$,$p_3 = (0, 5)$, $p_4 = (1, 1)$, $p_5 = (3, 1)$ and $p_6 = (1, 3)$.

i) 🖳 Are there weights w_{ij} such that this embedding is in equilibrium condition? (Find them or explain why they cannot exist.)

ii) 🖳 Compute and draw the equilibrium embedding for the case when all the weights are $\omega_e = 1$.

(10 points)

Exercise 3. Consider the following embedding of a planar graph G



where $p_1 = (0, 0)$, $p_2 = (5, 0)$,$p_3 = (0, 5)$, $p_4 = (1, 1)$, $p_5 = (3, 1)$, $p_6 = (1, 3)$ and $p_7 = (\frac{1}{2}.3)$.

Define its **space of liftings** $\mathcal{H}$ as

$$\mathcal{H} := \{(h_4, \dots, h_7) \in \mathbb{R}^4_{\geq 0} \ : \ \mathrm{graph}(\mathsf{conv}(\{(p_i, h_i) \ : \ 1 \leq i \leq 7\})) = G\},$$

where $h_1 = h_2 = h_3 = 0$.

That is, the set of choices of $h_i \geq 0$ such that lifting each point p_i to (p_i, h_i) gives the set of vertices of a polytope with the desired graph.

i) Prove that its closure $\overline{\mathcal{H}}$ is a convex polyhedral cone.

ii) Consider $\overline{\mathcal{H}} \cap \{h_5 = 1\}$. Show that it is a polytope and plot it.

iii) For each vertex v of $\overline{\mathcal{H}} \cap \{h_5 = 1\}$, consider the associated lifting of G. Compute these liftings and draw the graphs of the polytopes that you obtain.

(10 points)

Exercise 4. i) Let G be a k-connected graph and let S be a subset of its vertices of size $|S| \geq k$. Let G' be the graph obtained from G by adding one additional vertex connected to each $v \in S$. Prove that G' is k-connected.

ii) A subdivision of a graph G is any graph obtained by replacing edges of G by paths. Show that the graph of every 3-polytope contains subdivision of K_4.

iii) Show that the graph of every 4-polytope contains subdivision of K_5. [Hint: Look up Menger's Theorem.]

iv) Conclude that $G(P)$ is not planar for $\dim P \geq 4$. [Hint: Look up Kuratowski's Theorem.]

v) [Bonus: Show that the graph of every d-polytope contains subdivision of K_{d+1}.]

(10+3 points)