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Discrete Geometry I

Homework # 7 — due December 3rd

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be full-dimensional nonempty polytopes containing the origin in its interior. Find necessary and sufficient conditions for ...

i) ... $P \times Q$ being simple.

ii) ... $P \times Q$ being simplicial.

iii) ... $P \times Q$ being k-neighborly.

iv) ... $P \oplus Q$ being simple.

v) ... $P \oplus Q$ being simplicial.

vi) ... $P \oplus Q$ being k-neighborly.

vii) ... $P * Q$ being simple.

viii) ... $P * Q$ being simplicial.

ix) ... $P * Q$ being k-neighborly.

(10 points)

Exercise 2. Let $F \subset \mathbb{R}^d$ be a polytope and H a hyperplane.

i) Show that if $F \not\subseteq H$, then either $F \cap H$ is a proper face of F, or relint$(F) \cap H \neq \emptyset$.

ii) Show that if $F \not\subseteq H$ and relint$(F) \cap H \neq \emptyset$, then $\dim F \cap H = \dim F - 1$.

Let $P \subset \mathbb{R}^d$ be a d-polytope. For $v \in V(P)$, let H be a hyperplane strictly separating v and $P' =$ conv$(V(P) \setminus \{v\})$ and let $P_v = P \cap H$.

iii) Show that there is an inclusion-preserving bijection between $(k-1)$-faces of P_v and k-faces of P containing v for all $k = 0, \ldots, d$. In particular, $[v, P]_{\mathcal{L}(P)} \cong \mathcal{L}(P_v)$.

iv) For every proper face F of P, show that there is a polytope P_F such that $[F, P]_{\mathcal{L}(P)} \cong \mathcal{L}(P_F)$.

(10 points)

Exercise 3. i) Show that the Boolean lattice $\mathcal{B}_n = (2^{[n]}, \subseteq)$ is Eulerian.

ii) Show that for every nonempty polytope P, $\mathcal{L}(\mathrm{pyr}(P))$ is Eulerian if and only if $\mathcal{L}(P)$ is.

iii) Show that for every nonempty polytope P, $\mathcal{L}(\mathrm{prism}(P))$ is Eulerian if and only if $\mathcal{L}(P)$ is.

iv) Find a graded, atomic and coatomic lattice with the diamond property and rank $\mathrm{rk}(\hat{1}) \in \{3, 4\}$ that is not the face lattice of a polytope.

v) Can you find an example that is Eulerian?

(10 points)

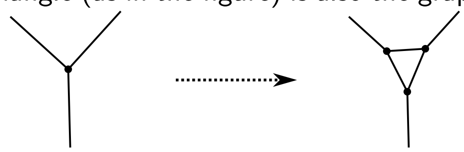
Exercise 4. For a polyhedron Q, we define the graph of Q as $G(Q) = (V, E)$ with $V = V(Q)$ and $E = \{uv : [u, v] \subset Q \text{ edge}\}$.

i) Is it true that every $G(Q)$ is connected for every pointed d-polyhedron?

ii) What is the maximal $k = k(d)$ such that every d-polyhedron Q is at least k-connected?

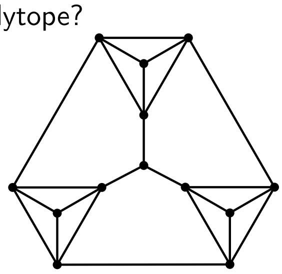
Let P be a 3-polytope with graph $G = G(P)$.

iii) Prove that if G has a node v of degree 3, then the graph G' obtained by replacing v by a triangle (as in the figure) is also the graph of a 3-polytope.



What is the polar operation (in terms of the graph of the polar polytope)?

iv) Prove that the following graph is the graph of a polytope (find vertex coordinates and check with SAGE that you obtain the correct graph). Can it be the graph of a 4-polytope?



(10 points)

Bonus Exercise. A graph is **Hamiltonian** if it contains a cycle that visits every node without repeating any edge. Can you find a 3-polytope whose graph is not Hamiltonian?

(+ 5 points)