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Discrete Geometry I

Homework # 6— due November 26th

Please mark **two** of the exercises (but try to solve all of them). State who wrote the solution.

Exercise 1. Let $P \subset \mathbb{R}^d$ with 0 in the interior and $F \subseteq P$ a face. The face associated to F is

$$F^\diamond \ := \ \{c \in P^\triangle : c^t x = 1 \text{ for all } x \in F\}$$

Prove the following properties.

i) $F^\diamond$ is a face of $P^\triangle$.

ii) $(F^\diamond)^\diamond = F$.

iii) If G is a face of P with $F \subsetneq G$, then $G^\diamond \subsetneq F^\diamond$.

iv) $\dim F + \dim F^\diamond = \dim P - 1$.

(10 points)

Exercise 2. i) Let $\mathcal{P}$ be a finite and bounded poset. Show that if $\mathcal{P}$ is a meet-semilattice then it is also and a join-semilattice. Such a poset is called a **lattice**.

ii) A natural number n is called squarefree if in a prime factorization of n no prime factor occurs more than once. Show that for $a < b \in \mathcal{D}$ with $\frac{b}{a}$ squarefree, the interval $[a, b]_\mathcal{D}$ is isomorphic to the face lattice of a d-simplex for some d.

iii) Let $\mathcal{A}_n = \widehat{\mathcal{A}}_n \cup \{\hat{0}\}$ be the poset of partial backetings together with a minimal element. Show that $\mathcal{A}_n$ is a lattice.
Show that for every $x \neq \hat{0}$, the interval $[x, \hat{1}] \subset \widehat{\mathcal{A}}_n$ is isomorphic to a Boolean lattice $\mathcal{B}_k$ for some k.

iv) Can you find a polytope P such that $\mathcal{L}(P) \cong \mathcal{A}_4$?
[Bonus: How about $\mathcal{A}_5$?]

(10+5 points)

Exercise 3. Let P be a d-polytope with vertices $v_1, \ldots, v_n$ and facets $F_1, \ldots, F_m$. We define $\mathcal{I}(P) = \{(i, j) \in [n] \times [m] : v_i \in F_j\}$.

i) Prove that for a proper face $G \subseteq P$,

$$V(G) = \bigcap \{V(F_i) : G \subseteq F_i, i = 1, \ldots, m\}.$$

ii) For a given set of vertices $J \subseteq [n]$, how can you check that there is a face F with $V(F) = \{v_i : i \in J\}$ only using $\mathcal{I}(P)$?

iii) Let Q be a polytope. Show that knowing $\mathcal{I}(P)$ and $\mathcal{I}(Q)$ suffices to check if $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are isomorphic.

iv) [Bonus: Write a SAGE function that receives a list of vertex-facet incidences and outputs the face lattice (as a Poset). Check the correctness of your function with the cyclic polytopes $\mathrm{Cyc}_2(6)$, $\mathrm{Cyc}_3(6)$ and $\mathrm{Cyc}_4(6)$. (Check for poset isomorphism.)]

(10+5 points)