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Discrete Geometry I

Homework # 5— due November 19th

Please mark **two** of the exercises (but try to solve all of them).

Exercise 1. [This is a continuation of Exercises 1 & 2 on Homework # 4.]
How to compute a $\mathcal{H}$-representation from a $\mathcal{V}$-representation.

For a set $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$ consider the polyhedron

$$\Delta(V) := \left\{ (x, \lambda) \in \mathbb{R}^d \times \mathbb{R}^n \ : \ \lambda \geq 0, \ x = \sum_i \lambda_i v_i, \ \sum_i \lambda_i = 1 \right\}.$$

i) Show that the projection of $\Delta(V)$ onto the x-coordinates equals conv(V). Using Exercise 1 on Homework #4, design an algorithm to compute an $\mathcal{H}$-representation of conv(V).

ii) Implement this algorithm in SAGE and compute an inequality description of the 3-polytope with vertices $\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$.

(10 points)

Exercise 2. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be polytopes with 0 in the interior. Recall that the join of P and Q is the polytope

$$P * Q \ = \ \text{conv}\{(p, 0, 1), (0, q, -1) : p \in P, q \in Q\} \subset \mathbb{R}^{d+e+1}$$

i) Show that $0 \in \text{int}(P * Q)$.

ii) Show that $(P * Q)^{\triangle}$ and $P^{\triangle} * Q^{\triangle}$ are affinely equivalent.

iii) Given $P = \{x : Ax \leq b\}$ and $Q = \{y : By \leq c\}$, both bounded and containing 0 in the interior, give an inequality description of $P * Q$.

(10 points)

Exercise 3. Let $P = \text{conv}(v_1, \dots, v_n)$. Prove that if $x = \sum_i \lambda_i v_i$ with $\sum_i \lambda_i = 1$ and $\lambda_i > 0$ for all i, then $x \in \text{relint}(P)$. Is the converse also true (i.e. if $x = \sum_i \lambda_i v_i$ with $\sum_i \lambda_i = 1$ and $x \in \text{relint}(P)$, then $\lambda_i > 0$ for all i)?

(10 points)

Exercise 4. i) Let $T(x) = Ax$ be an invertible linear transformation. For nonempty $P \subseteq \mathbb{R}^d$ show that $T(P)^{\triangle} = T^*(P^{\triangle})$ where $T^*(x) = (A^t)^{-1}x$.

ii) Let $B_d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ be the d-dimensional unit ball. Show that $B_d^{\triangle} = B_d$.

iii) Let $K \subseteq \mathbb{R}^d$ be a convex set. Show that $K^{\triangle}$ is bounded if and only if $0 \in \text{int}(K)$.

(10 points)