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Discrete Geometry I

Homework # 3— due November 5th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday. Homework sheets and additional information (including corrections) will be announced on the mailinglist. If you haven't signed up already, please do that at

`https://lists.fu-berlin.de/listinfo/dg1`

Exercise 1. Let $a = (a_1 \geq a_2 \geq \cdots \geq a_d)$, the **generalized permutahedron** is defined by

$$\Pi_{d-1}(a) \;=\; \text{conv}\{(a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(d)}) : \pi \text{ permutation of } [d]\ \}$$

i) Write a function that, given a as input, constructs the generalized permutahedron $\Pi_{d-1}(a)$.

ii) Prove that if $a_1 \geq a_2 \geq \cdots \geq a_d$ and $b_1 \geq b_2 \geq \cdots \geq b_d$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_d b_d \geq a_1 b_{\pi(1)} + a_2 b_{\pi(2)} + \cdots + a_d b_{\pi(d)}$$

for every permutation π of $[d]$. When is the inequality strict?

iii) Assume that $a = (a_1 > a_2 > \cdots > a_d)$. For a permutation π, denote by $\pi(a) = (a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(d)})$. Show that $[a, \pi(a)] = \text{conv}\{a, \pi(a)\}$ is an edge of $\Pi_{d-1}(a)$ if π is an **adjacent transposition**, that is, $\pi(a) = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_d)$ for some $1 \leq i < d$.

iv) Using (iii), write a function that makes the graph of $\Pi_{d-1}(a)$, $a = (a_1 > a_2 > \cdots > a_d)$. (Without computing the polytope.) Check for $3 \leq d \leq 5$ that the graph you generate with this function is isomorphic to the graph of the polytope $\Pi_{d-1}(a)$ computed with the function from (i) (for some $a = (a_1 > a_2 > \cdots > a_d)$).

(10 points)

Exercise 2. Let $G = (V, E)$ be a graph with n vertices $V = \{v_1, \ldots, v_n\}$. A **stable (or independent) set** of G is a subset $S \subseteq V$ such that there is no edge of G between two elements of S.

The indicator vector of a stable set S is the vector $\mathbf{1}(S) \in \mathbb{R}^n$ with coordinates

$$\mathbf{1}(S)_i = \begin{cases} 1 & \text{if } v_i \in S \\ 0 & \text{otherwise.} \end{cases}$$

The **stable set polytope** of a graph G is

$$\mathrm{STAB}(G) = \text{conv}(\{\mathbf{1}(S) \ : \ S \text{ stable set of } G\}).$$

i) Prove that $\mathrm{STAB}(G)$ is of dimension n and that $\{\mathbf{1}(S) : S \subseteq V \text{ stable}\}$ are its vertices.

ii) Write a SAGE function that computes $\mathrm{STAB}(G)$ for an input graph G.

iii) What is $\mathrm{STAB}(G)$ for the complete graph K_n? Compute the f-vectors of $\mathrm{STAB}(C_m)$ for the m-cycles $m = 3, 4, 5, 6$.

iv) Show that

$$\mathrm{STAB}(G) \subseteq \{x \in \mathbb{R}^n : x \geq 0, x_u + x_v \leq 1 \text{ for all } uv \in E\}.$$

Find an example where the inclusion is strict.

v) A subset $K \subseteq V$ is a **clique** if $uv \in E$ for all $u, v \in K$, $u \neq v$. Show that you can extend the inequality formulation of (iv) using cliques. Can you find an example where the inclusion is still strict?
[Hint : Try the examples from (iii)].

vi) For $W \subseteq V$, the induced subgraph $G[W]$ has vertices W and edges $\{uv \in E : u, v \in W\}$. Show that $\mathrm{STAB}(G[W])$ is a face of $\mathrm{STAB}(G)$.

(10 points)

Exercise 3. Let $C = \text{cone}\{u_1, \ldots, u_n\} \subseteq \mathbb{R}^d$ be a **convex polyhedral cone**. That is,

$$C = \{x \in \mathbb{R}^d \ : \ x = \sum \lambda_i u_i, \ \lambda_i \geq 0\}.$$

We say that C is **pointed** if it does not contain a whole line through the origin. That is, there is no $v \in \mathbb{R}^d$ such that $\lambda v \in C$ for all $\lambda \in \mathbb{R}$.

i) Prove that C is pointed if and only if 0 is a face, that is, $C \cap H = \{0\}$ for some supporting hyperplane $H = \{x : c^t x = 0\}$.

ii) With this hyperplane show that $C_p = C \cap (p + H)$ is a polytope for any $p \in C \setminus \{0\}$.

iii) Deduce that if $u_i \neq \lambda u_j$ for all $\lambda \in \mathbb{R}$ and $i \neq j$, then there is a unique inclusion-minimal subset $U' \subseteq U$ such that $C = \text{cone}(U')$.

(10 points)

Exercise 4. Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be full-dimensional polytopes containing the origin in their interior (that is $0 \in H^- \setminus H$ for every supporting hyperplane H). The **free sum** of P and Q is the polytope $P \oplus Q$ defined by

$$P \oplus Q := \text{conv}\left(\left\{(p, 0) \in \mathbb{R}^{d+e} \ : \ p \in P\right\} \cup \left\{(0, q) \in \mathbb{R}^{d+e} \ : \ q \in Q\right\}\right).$$

i) Express the d-crosspolytope $C_d^{\triangle} = \{x \in \mathbb{R}^d \ : \ \sum |x_i| \leq 1\}$ as a free sum of d segments.

ii) Let P be a polytope that contains the origin in the interior and let F be a proper face of P. Prove that there is a supporting hyperplane H_a for F of the form $H_a = \{x \in \mathbb{R}^d \ : \ a^t \cdot x = 1\}$ such that for every $x \in P$, $a^t \cdot x \leq 1$.

iii) Let F and G be proper (maybe empty) faces of P and Q, respectively. Prove that the set

$$V_{F,G} = \{(p, 0) \ : \ p \in V(P) \cap F\} \cup \{(0, q) \ : \ q \in V(Q) \cap G\}$$

is the set of vertices of a proper face of $P \oplus Q$.

iv) Prove that every proper face of $P \oplus Q$ has a set of vertices of the form $V_{F,G}$ for some proper faces F and G of P and Q, respectively.

v) Express $f_k(P \oplus Q)$, $-1 \leq k \leq d + e$, in terms of the face numbers of P and Q.

(10 points)