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# Discrete Geometry I

## Homework # 2— due October 29th

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a ◻ are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday. Homework sheets and additional information (including corrections) will be announced on the mailinglist. If you haven't signed up already, please do that at

`https://lists.fu-berlin.de/listinfo/dg1`

**Exercise 1.** Let $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ be a point configuration affinely spanning $\mathbb{R}^d$ (i.e. $\mathrm{aff}(V) = \mathbb{R}^d$). Let $\mathcal{H}$ be the collection of hyperplanes spanned by $V$.

i) Show that there is at least one hyperplane $H \in \mathcal{H}$ that is supporting for $P = \mathrm{conv}(V)$.

ii) Show that every $d$-polytope has a $(d-1)$-face.
[Bonus (+3): Show that every $d$-polytope has at least $d+1$ $(d-1)$-faces.]

**(10+3 points)**

**Exercise 2.** Let $P$ be a $d$-polytope and $1 \leq k \leq d$. We say that $P$ is $k$-**neighborly** if $\mathrm{conv}(V')$ is a face of $P$ for all $V' \subseteq V(P)$ with $|V'| = k$, where $V(P)$ is the set of vertices of $P$.

i) Show that if $P$ is $k$-neighborly, then $P' = \mathrm{conv}(V')$ is also $k$-neighborly for all $V' \subseteq V(P)$.

ii) Show that if $P$ is $k$-neighborly then it is $\ell$-neighborly for every $\ell \leq k$.

iii) Write a SAGE/polymake function that computes the neighborliness of a polytope and verify that the cyclic $d$-polytope with $n$ vertices is $\lfloor \frac{d}{2} \rfloor$-neighborly for all $2 \leq d \leq 10$ and $d+1 \leq n \leq 2d$.

**(10 points)**

**Exercise 3.** For $d < n$ and $r_1 < r_2 < \cdots < r_n$ let

$$C = \mathrm{Cyc}_d(r_1, r_2, \ldots, r_n) = \mathrm{conv}\{\gamma_d(r_i) : i = 1, 2, \ldots, n\}$$

be a cyclic $d$-polytope. (Recall that $\gamma_d(t) = (t, t^2, \ldots, t^d)^t$.)

i) Let $I \subseteq [n]$. An element $j \notin I$ is an **odd** or **even gap** if the number $|\{i \in I : i < j\}|$ is odd or even, respectively.
For $I \subseteq [n]$ with $|I| = d$ show that

$$F_I = \mathrm{conv}\{\gamma_d(r_i) : i \in I\}$$

is a facet of $C$ if and only if all gaps of $I$ are either all even or odd.

[Bonus (+3): What is the number (closed formula!) of facets $f_{d-1}(C)$? What is the number of $k$-faces?]

ii) Consider a **lifting** of a cyclic polytope $\mathrm{Cyc}_d(n)$ with $n > d+1$. That is a $(d+1)$-polytope $P$ with $n$ vertices of the form $v_i = (\gamma_d(r_i), \alpha_i)$ for some $r_i, \alpha_i \in \mathbb{R}$. Prove that $P$ is $\lfloor \frac{d}{2} \rfloor$-neighborly.

iii) Generate 5-dimensional polytopes with 10 vertices that are 2-neighborly (that is, their graph is the complete graph) and have a facet with $k$ vertices for $k \in \{6, \ldots, 9\}$. Is any of these affinely isomorphic to a cyclic polytope?

**(10+3 points)**

**Exercise 4.** i) Given non-empty polytopes $P_1$ and $P_2$ show that for every pair of faces $F_1 \subseteq P_1, F_2 \subseteq P_2$ their product $F_1 \times F_2$ is a face of $P_1 \times P_2$. And conversely, for every non-empty face $F$ of $P_1 \times P_2$ there are unique faces $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$ such that $F = F_1 \times F_2$.

ii) Let $\pi : P \to Q$ be an affine projection of polytopes (that is, there is an affine map $\pi : \mathbb{R}^d \to \mathbb{R}^e$ such that $\pi(P) = Q$). Show that $\pi^{-1}(F) \cap P$ is a face of $P$ for every face $F \subseteq Q$.

iii) Given non-empty polytopes $P_1$ and $P_2$ show that if $F \subseteq P_1 + P_2$ is a face, then $F = F_1 + F_2$ for some faces $F_i \subseteq P_i$. Show that $F_1$ and $F_2$ are unique.

**(10 points)**