Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Dr. Arnau Padrol

Discrete Geometry I

Homework # 1— due October 22nd

You can write your solution to the homeworks in **pairs**. Please try to solve *all* problems. This will deepen the understanding of the material covered in the lectures. You are welcome to ask (in person or email) for additional **hints** for any exercise. Please think about the exercise before you ask. Please mark **two** of your solutions. Only these will be graded. The problems marked with a 🗊 are **mandatory**. You can earn **20 points** on every homework sheet. You can get extra credit by solving the bonus problems. **State** who wrote up the solution. You have to hand in the solutions **before** the recitation on Wednesday. Homework sheets and additional information (including corrections) will be announced on the mailinglist. If you haven't signed up already, please do that at

`https://lists.fu-berlin.de/listinfo/dg1`

Some of the exercises require computations, they are marked with a 🖱. We will mainly use the computer programs `sage` and `polymake` for these.

- `sage` is a mathematical software that covers several aspects of mathematics, polytopes among them. It is called on UNIX-computers at FU by the command line

  `/import/sage-6.2/sage`

  It can be also used online at

  `http://sagenb.org/`

  More information (installation, tutorial, etc.) can be found at

  `http://www.sagemath.org/`

  The manual page concerning polyhedra is

  `http://www.sagemath.org/doc/reference/geometry/sage/geometry/polyhedron/constructor.html`

- `polymake` allows to analyze combinatorial properties of polytopes. It is called on UNIX-computers at FU by the command line

  `/import/polymake/bin/polymake`

  More information (installation, tutorial, etc.) can be found at

  `http://polymake.org/`

When handing in computational exercises, you have to either attach a **printed** version of your code, or send it by **email**.

Exercise 1. Recall that the ℓ_∞-norm is $\|x\|_\infty = \max_i |x_i|$. The d-dimensional **cube** is the set

$$C_d = \{x \in \mathbb{R}^d \ : \|x\|_\infty \leq 1\}$$

i) Show that C_d is a polytope.

A **zonotope** $Z \subset \mathbb{R}^d$ is the Minkowski sum of a finite collection of segments (in $\mathbb{R}^d$). That is, there are $z_1, \ldots, z_m$ such that

$$Z = [-z_1, z_1] + [-z_2, z_2] + \cdots + [-z_m, z_m].$$

ii) Show that C_d is a zonotope.

iii) Show that every zonotope is an affine projection of a cube.

(10 points)

Exercise 2. i) Prove that, if P and Q are polytopes, then their Minkowski sum $P + Q$ is also a polytope.

[Bonus (+3): Show that the reverse is also true: if the Minkowski sum of two convex sets is a polytope, then both summands are polytopes.]

ii) Write a sage function that computes the Minkowski sum of two polytopes.

iii) Let P and Q be an n-gon and an $m-$gon in $\mathbb{R}^2$. Show that

$$\max\{n, m\} \leq |\text{vertices}(P + Q)| \leq n + m$$

Prove that both bounds can be attained.

iv) How about two polygons embedded in 3-space? For each $k \in \{3, \ldots, 9\}$ find (experimentally) two triangles embedded in $\mathbb{R}^3$ whose Minkowski sum has k vertices.

[Bonus (+3): What is the maximal number of vertices for triangle + quadrilateral in $\mathbb{R}^3$?]

(10+6 points)

Exercise 3. For a family of vectors $v_1, \ldots, v_k \in \mathbb{R}^d$, consider the set

$$B(v_1, \ldots, v_k) = \left\{ x \in \mathbb{R}^d \ : \ \sum_{i=1}^{k} |\langle x, v_i \rangle| \leq 1 \right\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product.

i) Prove that $B(v_1, \ldots, v_k)$ is always convex.

ii) When is $B(v_1, \ldots, v_k)$ bounded? (Find conditions on the v_i's.)

iii) Prove that $B((1,0), (0,1), (1,1))$ is a polygon. What are its vertices?

(10 points)

Exercise 4. The **polyhedral cone** spanned by $v_1, \ldots, v_n \in \mathbb{R}^d$ is the set

$$\text{cone}(v_1, \ldots, v_n) := \{x \in \mathbb{R}^d \ : \ x = \mu_1 v_1 + \cdots + \mu_n v_n : \mu_1, \ldots, \mu_n \geq 0\}$$

i) Prove that every linear subspace is a polyhedral cone.

ii) Let L is a k-dimensional linear subspace. What is the minimal number of generators $\{v_1, \ldots, v_n\}$ such that $L = \text{cone}\{v_1, \ldots, v_n\}$?

iii) Characterize those sets $\{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ such that

$$\text{conv}\{v_1, \ldots, v_n\} = \text{aff}\{v_1, \ldots, v_n\} \cap \text{cone}\{v_1, \ldots, v_n\}.$$

(10 points)