Freie Universität Berlin SS2014
Institut für Mathematik
Prof. Raman Sanyal
Katharina Jochemko

Combinatorial Reciprocity Theorems

Homework # 9 — due July 7, **before** the lecture

Exercise 1. Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional simplex with *rational* vertices $v_0, \dots, v_d$.

i) Show that the Ehrhart function $\mathrm{ehr}(P, n)$ is a quasipolynomial of degree d.

ii) Let the vertex coordinates be $v_i = (\frac{a_{i1}}{b_{i1}}, \dots, \frac{a_{id}}{b_{id}})$. Show that the period of the quasipolynomial divides the least common multiple of the denominators b_{ij} for $i = 0, \dots, d, j = 1, \dots, d$.

(6 points)

Exercise 2. Let $d(n)$ denote the number of partitions of n into distinct parts (i.e. no part is used more than once), and let $o(n)$ be the number of partitions of n into odd parts (i.e. every part is an odd integer). Compute the generating functions of $d(n)$ and $o(n)$ and show that they are equal (and thus $d(n) = o(n)$ for all positive integers n).

(6 points)

Exercise 3. Let $f(n)$ and $g(n)$ be a quasi-polynomials.

i) Show that $r(n) = \sum_{i=0}^{n} f(i)$ is a quasi-polynomial.

ii) More generally, show that the *convolution*

$$(f * g)(n) \sum_{i=1}^{n} f(i) g(n-i)$$

is a quasi-polynomial.

iii) Bonus: What can be said about degree and period of $f * g$ if degree and period of f and g are given?

(3+3+3 points)

Exercise 4. Bonus: Consider

$$\frac{1}{9899} = 0.00010203050813213455...$$

Why do the Fibonacci numbers 1, 2, 3, 5, 8, 13,... appear?

(3 points)