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Combinatorial Reciprocity Theorems

Homework # 8 — due June 30, **before** the lecture

Exercise 1. i) Let $\sum_{n\geq 0} f(n)t^n = \frac{p(t)}{q(t)}$ be a rational generating function with $q(t) = 1+\alpha_1 t + \cdots + \alpha_d t^d$ and $\deg p \geq \deg q$. Show that there is a finite *exceptional set* $E \subset \mathbb{N}$ and a function $g : \mathbb{N} \to \mathbb{C}$ such that $g(n) = 0$ for $n \in \mathbb{N} \setminus E$ such that

$$f(n+d) \ + \ \sum_{i=1}^{d} \alpha_i f(n+d-i) \ = \ g(n+d) \ + \ \sum_{i=1}^{d} \alpha_i g(n+d-i)$$

for all $n \in \mathbb{N}$.

ii) Bonus: Consider $\mathbb{N}^n$ as a poset with the componentwise partial order and let $I \subset \mathbb{N}^n$ be a *filter*, i.e. if $a \succeq b \in I$ then $a \in I$. Show that there are $a^1, a^2, \ldots, a^M \in \mathbb{N}^n$ such that

$$I \ = \ \left\{ a \in \mathbb{N}^n \ : \ a \succeq a^i \text{ for some } i = 1, \ldots, M \right\}.$$

[Hint: Consider projections $(b_1, \ldots, b_n) \mapsto (b_1, \ldots, b_{n-1})$.]

iii) Let $I \subseteq \mathbb{N}^n$ be a filter and let

$$f(k) \ = \ \#\{b \in I : b_1 + b_2 + \cdots + b_n = k\}.$$

Show that $f(k)$ is a polynomial for $k \gg 0$ sufficiently large.
[Hint: This is easy if $M = 1$ and $I = \{a \succeq a^1\}$. In case of more generators the existence of joins and inclusion-exclusion might help.]
Bonus: Can you say what the exceptional set is?

(2+4+4+2 points)

Exercise 2. Let $A \in \mathbb{R}^{d\times d}$ be a symmetric matrix. For fixed $i, j \in [d]$ show that there are $b_1, b_2, \ldots, b_d, \lambda_1, \ldots, \lambda_d \in \mathbb{R}$ such that for all $n \geq 1$

$$(A^n)_{ij} \ = \ b_1\lambda_1^n \ + \ b_2\lambda_2^n \ + \ \cdots \ + \ b_d\lambda_d^n.$$

(4 points)

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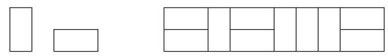
Exercise 3. Let $\{F_n\}_{n\geq 0}$ be the number of tilings of a rectangle of size $2 \times n$ with rectangles of size 2×1 and 1×2.



i) Compute $\sum_{n\geq 0} F_n t^n$.

ii) Show that
$$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - (-1)^n \frac{1}{\varphi^n}\right)$$
where $\varphi = \frac{1+\sqrt{5}}{2}$.

iii) Bonus: What does that all have to do with rabbits?

(3+3+1 points)

Exercise 4. For $k \geq 1$ fixed, let $f_k(n) = n^k$.

i) Show that there are numbers $\alpha_1, \ldots, \alpha_{k+1}$ such that
$$f_k(n+k+1) \;+\; \alpha_1 f_k(n+k) \;+\; \cdots \;+\; \alpha_{k+1} f_k(n) \;=\; 0.$$

ii) This implies that the generating function is rational
$$\sum_{n\geq 0} f_k(n) \;=\; \frac{p_k(t)}{q_k(t)}.$$
What is $q_k(t)$?

iii) Bonus: What is $p_k(t)$?
[Hint: Ask Google.]

iv) What is the rational generating function for
$$g_k(n) \;=\; \sum_{i=0}^{n} f_k(i)?$$

(1+1+3+2 points)

Exercise 5. Bonus: Let $f : \mathbb{N} \to \mathbb{C}$ and, for $N \in \mathbb{N}$, let $H_N(f)$ be the *Hankel determinant* $H_N(f) \;=\; \det(f(i+j-2))_{i,j\in[N]}$. E.g. for $N = 4$ this is
$$H_4(f) \;=\; \det\begin{pmatrix} f(0) & f(1) & f(2) & f(3) \\ f(1) & f(2) & f(3) & f(4) \\ f(2) & f(3) & f(4) & f(5) \\ f(3) & f(4) & f(5) & f(6) \end{pmatrix}$$
Show that $\sum_{n\geq 0} f(n)t^n$ is a rational generating function if and only if there is an N_0 such that $H_N(f) = 0$ for all $N \geq N_0$.
[Hint: Show sufficiency ($\Rightarrow$) first.]

(8 points)