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Combinatorial Reciprocity Theorems

Homework # 7 — due June 17, **before** the lecture

Exercise 1. A polyhedron $Q \subseteq \mathbb{R}^d$ is called *pointed* if it does not contain an affine line.

i) Show that a polyhedral cone $C \subseteq \mathbb{R}^d$ is pointed, if and only if for all $p \in \mathbb{R}^d$

$$p, -p \in C \quad \Rightarrow \quad p = 0.$$

ii) Let $Q \subseteq \mathbb{R}^d$ be a d-polyhedron and $U = \text{lineal}(Q)$. Pick $t \in Q$ and let $U^\perp$ be the orthogonal complement of U. Show that $\widehat{Q} = Q \cap (t + U^\perp)$ is a pointed polyhedron and that the map

$$F \;\mapsto\; \sigma(F) \;=\; F \cap (t + U^\perp)$$

is an order preserving bijection from $\mathcal{L}(Q)$ to $\mathcal{L}(\widehat{Q})$ that takes k-faces to faces of dimension $k - \dim(U)$.

iii) Show that $\chi(Q) = (-1)^{\dim(U)}\chi(\widehat{Q})$.

(1+4+1 points)

Exercise 2. The *f-vector* of a d-polyhedron Q is $f(Q) = (f_0, f_1, \ldots, f_d)$ with f_i being the number of faces of dimension i.

This exercise tries to get you in touch with software to do computations with polyhedra. You can "easily" do the computations using `polymake`[1], `SAGE`[2], etc. and hand in a print out of your session.

For $0 \leq k \leq d$ consider the $(d-1)$-polytope $\Delta(d,k) \subset \mathbb{R}^d$ given by the convex hull over all $p \in \{0,1\}^d$ such that p has exactly k ones.

Compute the f-vector of $\Delta(d, \lfloor \frac{d}{2} \rfloor)$ for $d = 4, 5, 6, 7$.

Bonus: Can you give a formula for $f_i(\Delta(d,k))$?

[Hint: What is the system of inequalities for $\Delta(d,k)$?]

(6+4 points)

Exercise 3. Let PC be the set of polyconvex sets in $\mathbb{R}^d$.

[1] www.polymake.org, Tutorials:http://polymake.org/doku.php/tutorial/start

[2] www.sagemath.org, Polyhedra in Sage: http://www.sagemath.org/doc/reference/geometry/sage/geometry/polyhedron/base.html

i) Show that there is a unique valuation $\phi\colon \mathrm{PC} \to \mathbb{R}$ such that for all relatively open polyhedra P with $L := \text{lineal}(P)$

$$\phi(P) = \begin{cases} (-1)^{\dim P - \dim L} & \text{if } P/L \text{ is bounded,} \\ 0 & \text{otherwise.} \end{cases}$$

Here, P/L is the orthogonal projection of P onto $L^{\perp}$.

ii) What is $\phi(Q)$ if Q is a polytope?

iii) What is $\phi(V)$ if V is an affine space?

iv) What is $\phi(\mathrm{hom}(Q))$ for Q a polytope?

v) Bonus: What is $\phi(Q)$ for an arbitrary closed polyhedron Q?

(4+1+1+2+4 points)